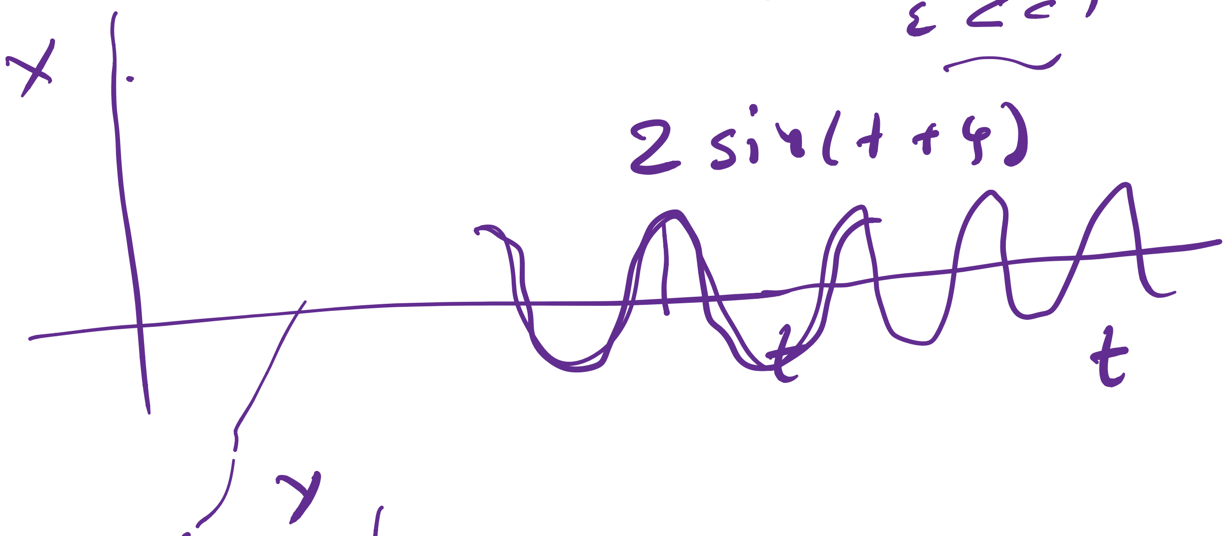
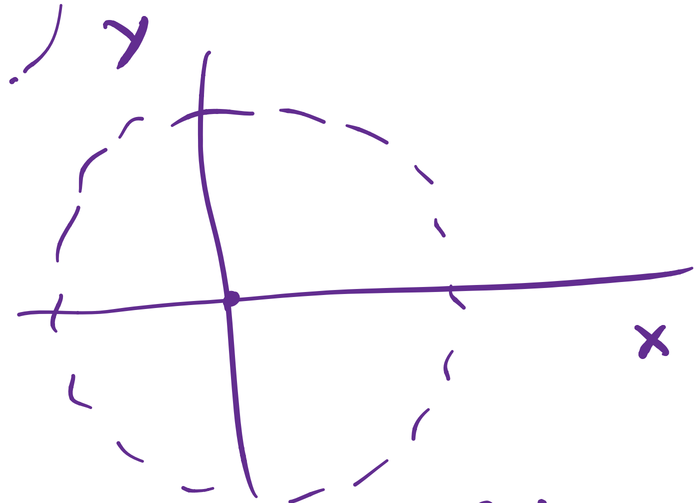
Τρίτη 8 Ιουνίου

$$\ddot{x} + x + \varepsilon \dot{x}(x^2 - 1) = 0$$

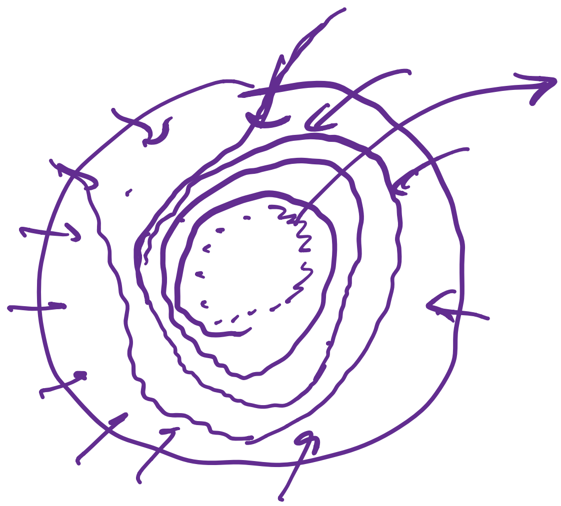
$$\varepsilon \ll 1$$





$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

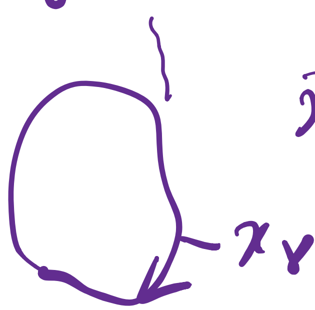
Poincaré-Bendixson



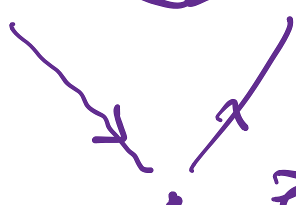
οριακοί

$\vec{x}_\gamma(t)$

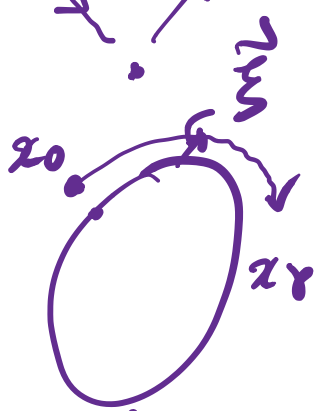
$$\dot{\vec{X}}_\gamma = \vec{f}(X_\gamma)$$



$$\vec{x}_\gamma(t+T) = \vec{x}_\gamma(t)$$



$$\frac{d\vec{x}}{dt} = \vec{v}(\vec{x})$$



$$\frac{d\vec{x}_\gamma}{dt} = \vec{v}(\vec{x}_\gamma)$$

$$\vec{\xi} = \vec{x} - \vec{x}_\gamma \qquad \frac{d\vec{\xi}}{dt} = \vec{v}(\vec{x}) - \vec{v}(\vec{x}_\gamma)$$

$$\frac{d\vec{\xi}}{dt} = \vec{v}(\vec{x}_\gamma + \xi) - \vec{v}(\vec{x}_\gamma)$$

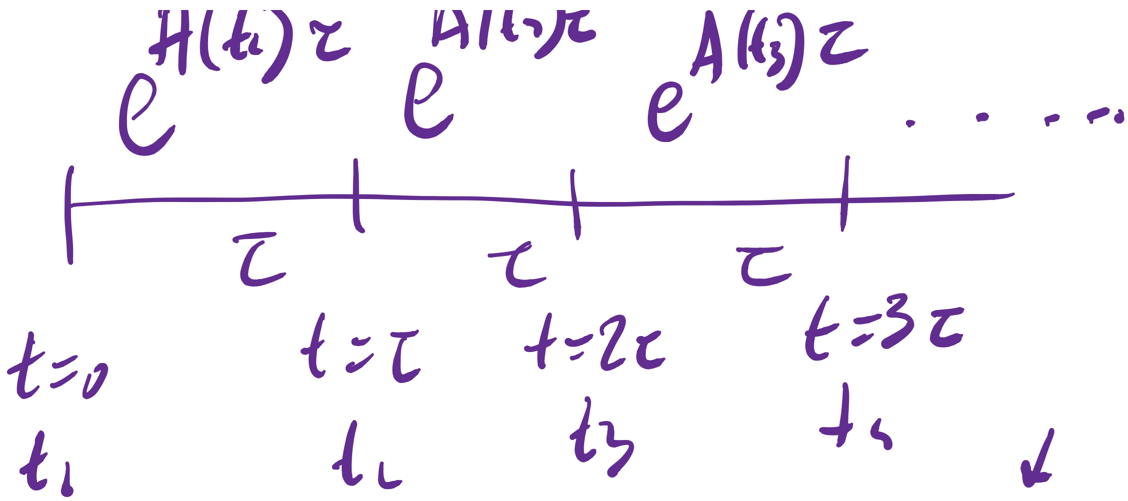
$$= (\vec{\xi}\cdot\nabla)_{x_\gamma}\vec{v} + O(\xi^2)$$

$$\frac{d\xi_i}{dt} = \sum_j \left.\frac{\partial v_i}{\partial x_j}\right|_{\vec{x}_\gamma(t)} \xi_j + O(\xi^2)$$

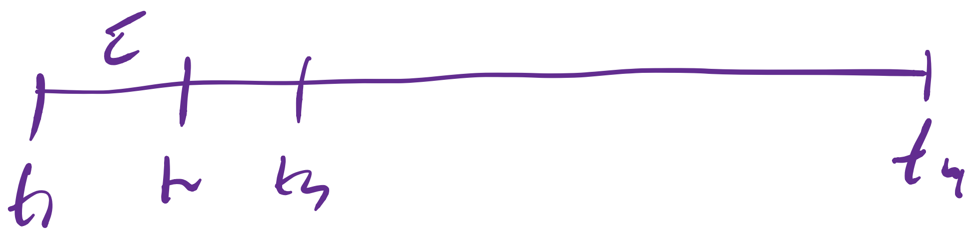
$$\sum_j \xi_j \frac{\partial}{\partial x_j} v_i = (\vec{\xi}\cdot\nabla)\vec{v}$$

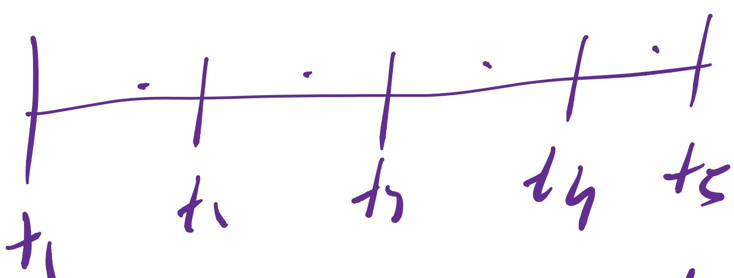
$$\frac{d\xi_i}{dt} = \underline{A_{ij}(t)}\,\xi_j$$

$$\frac{d\xi}{dt} = \underline{A(t)}\,\xi \qquad \xi(0) = \xi_0$$



$$\xi(t) = \Phi(t)\xi_0 = e^{A(t_{n-1})\tau} \cdots\cdots e^{A(t_2)\tau} e^{A(t_1)\tau} \xi_0$$





$$(n-1)\tau = t$$

$$\Phi(t,0) = \lim_{n\to\infty}\left(e^{A(t_{n-1})\tau} \cdots\cdots e^{A(t_1)\tau}\right)$$

$$(n-1)\tau = t\,, \quad \tau = \frac{t}{n-1}$$

$$\begin{cases} [A(t_i), A(t_j)] = 0 \\ \text{[illegible]} \\ e^{A_i\tau} e^{A_j\tau} = e^{(A_i + A_j)\tau} \end{cases}$$

δε ισχύει αυτό γενικά

$$\dot{\xi} = A(t)\,\xi$$

$$A(t+T) = A(t)$$

$\Phi(T)$

$$\xi(nT) = [\Phi(T)]^n \xi(0)$$

$$\Phi(T)$$

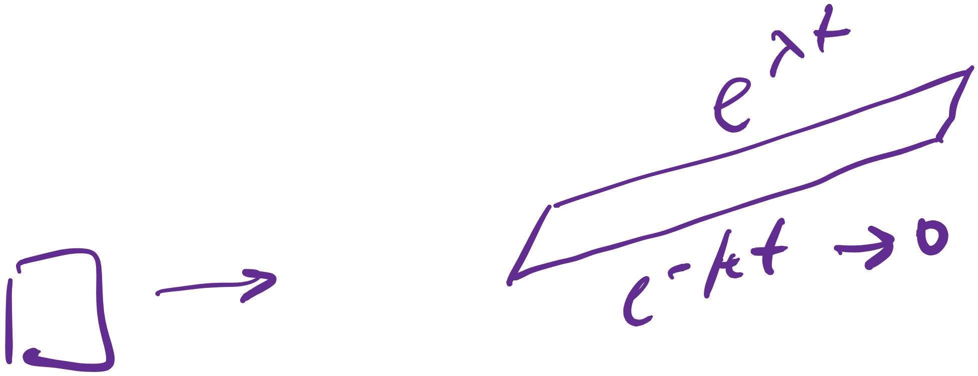
$$|\xi|^2 = \xi_1^2 + \xi_2^2$$

eig

$$|\xi| \alpha n \varepsilon \qquad \xi(nT) = \alpha^n \xi(0)$$

$$|\alpha| > 1$$

$$\xi(nT) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^n \xi(0)$$



$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = X \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} X^{-1}$$

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^4 = X \begin{pmatrix} \lambda_1^4 & 0 \\ 0 & \lambda_2^4 \end{pmatrix} X^{-1}$$

$$X \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cancel{X^{-1}} \cancel{X} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} X^{-1}$$

$$= X \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} X^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^2$$

$|\lambda_i| < 1$ τότε είναι ευσταθή

$\Phi(T)$ 2-διάστατη

$\lambda^2 - \mathrm{Trace}(\Phi)\lambda + \det\Phi = 0$

$\det \Phi =$

$$\Phi = \lim_{n\to\infty} e^{A_{n-1}\tau} \cdots e^{A_1\tau}$$

$$\det \Phi = \lim_{n\to\infty} \det\left(e^{A_{n-1}\tau}\right) \cdots \det\left(e^{A_1\tau}\right)$$

$$\det\left(e^{A\tau}\right) = e^{\mathrm{trace}(A)\tau}$$

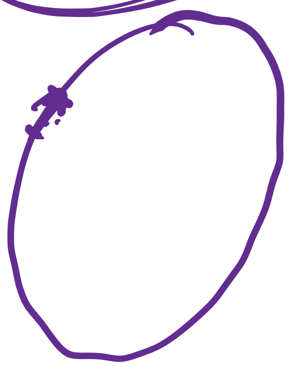
$$= \lim_{n\to\infty} e^{\mathrm{trace}(A_{n-1})\tau} \cdot e^{\mathrm{trace}(A_{n-2})\tau} \cdots e^{\mathrm{trace}(A_1)\tau}$$

$$= e^{\int_0^T \mathrm{trace}(A(s))\,ds}$$

$$\boxed{\lambda_1 \lambda_2 = e^{\int_0^T \mathrm{trace}(A(s))\,ds}}$$

$\lambda_1 = 1$!

$$\xi(T) = \underline{\Phi(T)}\,\xi(0)$$



$$\frac{d\vec{x}}{dt} = \vec{v}(\vec{x})$$

$$\frac{dv_i}{dt} = \sum_j \frac{\partial v_i}{\partial x_j}\frac{dx_j}{dt} = \sum \underbrace{\left.\frac{\partial v_i}{\partial x_j}\right|_{x_*}}_{A(t)} v_j$$

$$\frac{d\vec{v}}{dt} = A(t)\,\vec{v}$$

$$\lambda_2 = e^{\int_0^T \mathrm{trace}(A(s))\,ds}$$

$|\lambda_2| > 1$ απωθεί

αλλιώς έλκει ευσταθής

$$|\xi(nT)| = |\lambda_2|^n |\xi(0)|$$

$$\lim_{n\to\infty} \frac{\log \frac{|\xi(nT)|}{|\xi(0)|}}{nT} = \underline{\text{Lyapunov}}$$ <u>exponent</u>

$$\frac{\log|\lambda_2|}{T} = \text{exponent } \underline{\text{Lyapunov}}$$

$$\ddot{x} + x + \varepsilon \dot{x}(x^2 - 1) = 0$$

$$\underline{\varepsilon \ll 1}$$

$$x = 2\sin t$$
$$\dot{x} = 2\cos t$$

$$\dot{x} = y$$

$$\dot{y} = -x - \varepsilon y(x^2 - 1)$$

$x = 2\sin t \quad , \quad y = 2\cos t$

$$A(t) = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix}_{x_8} = \begin{pmatrix} 0 & 1 \\ -1 - 2\varepsilon xy & -\varepsilon(x^2 - 1) \end{pmatrix}$$

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 - 2\varepsilon xy & -\varepsilon(x^2 - 1) \end{pmatrix}$$

$x = 2\sin t \quad , \quad y = 2\cos t$

$$= \begin{pmatrix} 0 & 1 \\ -1 - 4\varepsilon \sin 2t & -\varepsilon(4\sin^2 t - 1) \end{pmatrix}$$

ϕ

$T = 2\pi$

$$\det \Phi(T) = e^{\int_0^{2\pi} -\varepsilon(4\sin^2 s - 1)\,ds}$$

$$= e^{-\varepsilon(4/2 - 1)2\pi}$$

$$= e^{-2\pi\varepsilon}$$

$$\lambda^2 - \mathrm{Trace}(\Phi)\lambda + e^{-2\pi\varepsilon} = 0$$

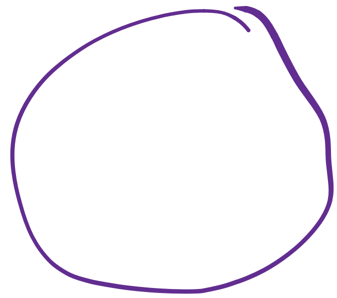
$\underline{\lambda_1 = 1}$ $\qquad$ $\lambda_2 = e^{-2\pi\varepsilon} < 1$

αν $\underline{\varepsilon > 0}$

Εκθέτης Lyapunov $\quad \dfrac{\log|\lambda_2|}{2\pi} = -\varepsilon$

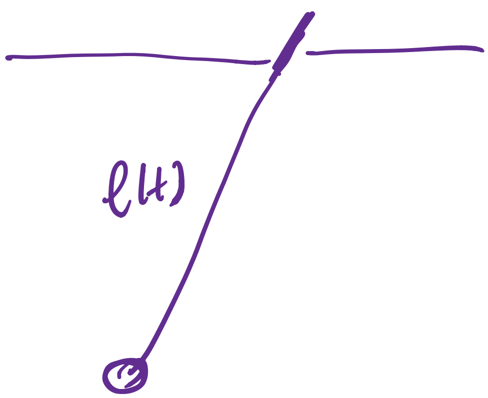
$t \to \infty$

$\sim \quad e^{-\varepsilon t}$



$e^{-(2\pi r) s}$

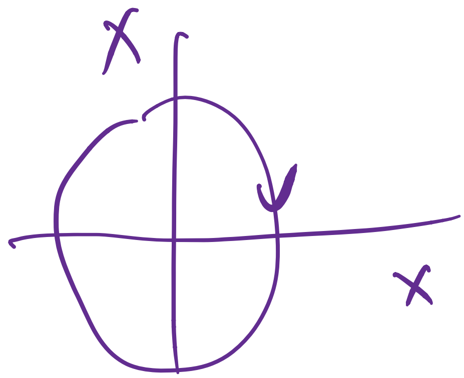
t

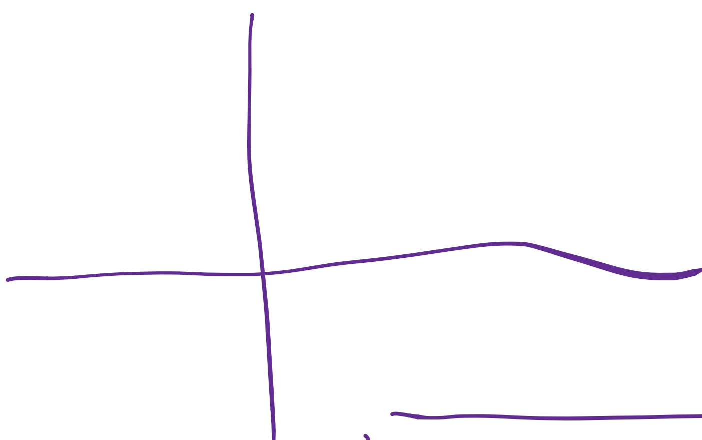


$$\ddot{x} + \omega^2 x = 0$$

$$\ddot{x} + \omega^2(t)\, x = 0$$

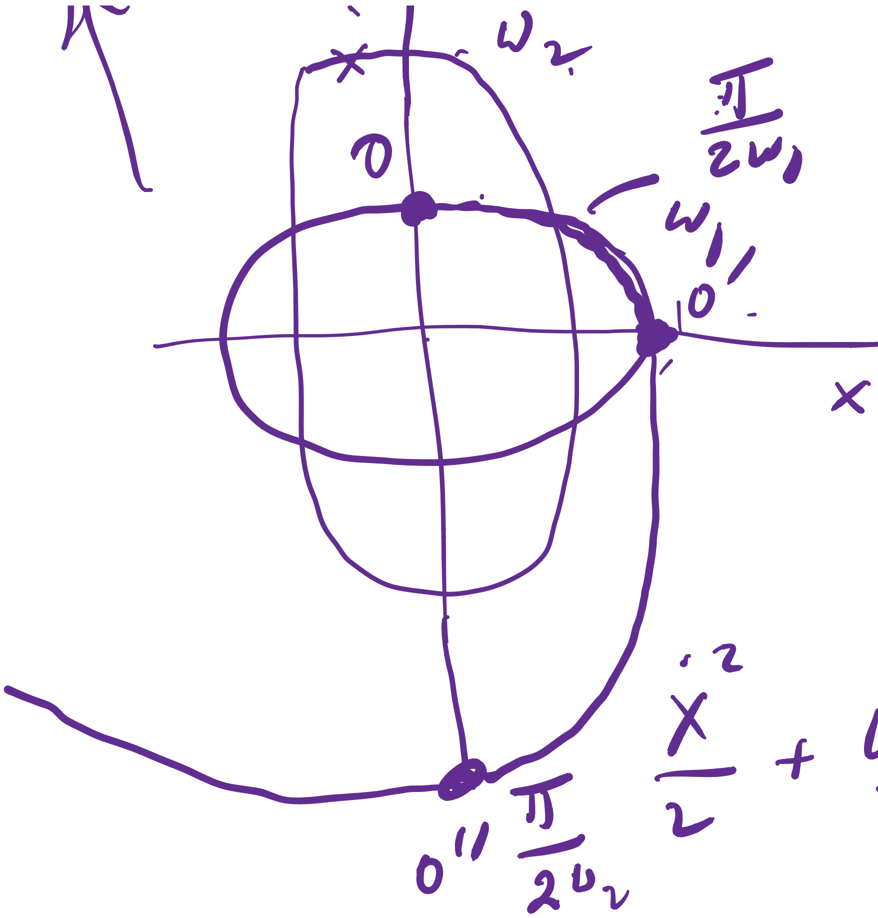
$$\boxed{\ddot{x} + \omega^2(1+\varepsilon \sin t)^2 x = 0}$$





$$\boxed{\begin{array}{l} \ddot{x} + \frac{1}{2} x = 0 \\ \ddot{x} + 4x = 0 \end{array}}$$

∞



$$\frac{\dot{x}^2}{2} + \frac{\omega_i^2}{2} x^2 = E$$

$$\dot{x} = y$$

$$\xi = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \dot{y} = -\omega^2(1+\varepsilon\sin\nu t)^2 x$$

$$\boxed{\frac{d}{dt}\xi = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2(1+\varepsilon\sin\nu t)^2 & 0 \end{pmatrix}}_{A(t)} \xi}$$

$$\boxed{\Phi(2\pi)} \quad \Phi(2\pi, \varepsilon, \omega)$$

$$\lambda^2 - \operatorname{trace}(\Phi)\lambda + \det\Phi = 0$$

1

$$\det\Phi = e^{\int_0^T 0 \cdot dt} = e^0 = 1$$

$$\det\left(e^{At}\right) = e^{\operatorname{trace}(A)t}$$

χαμιλτονιανό

$$\lambda_1 \lambda_2 = 1$$

εάν μιγαδικά $\lambda_1 = \lambda_2^*$

$$|\lambda_1| = |\lambda_2| = 1$$

ευστάθεια

μόνο αν $\lambda_1, \lambda_2 \in R$

[illegible]

$\omega \qquad \varepsilon$

$$\ddot{x} + \omega^2 x = 0 \qquad x(0) = 1, \quad \dot{x}(0) =$$

$\varepsilon = 0$

$\Phi(2\pi)$

$\Phi(2\pi)$

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega^2 x \end{aligned} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \cos\omega t \\ -\omega \sin\omega t \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} \frac{\sin\omega t}{\omega} \\ \cos\omega t \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

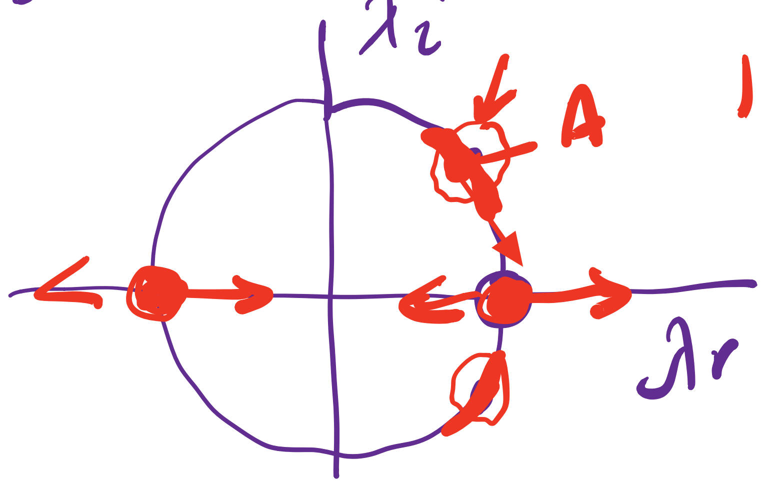
$$\Phi(2\pi, \underset{\substack{\uparrow \\ \varepsilon=0}}{0}, \omega) = \begin{pmatrix} \cos 2\pi\omega & \frac{\sin 2\pi\omega}{\omega} \\ -\omega \sin 2\pi\omega & \cos 2\pi\omega \end{pmatrix}$$

$\underline{\varepsilon=0}$

$$\lambda^2 - 2\cos 2\pi\omega\,\lambda + 1 = 0$$

$$\lambda = \cos 2\pi\omega \pm \sqrt{\cos^2 2\pi\omega - 1}$$

$$= \cos 2\pi\omega \pm i\sqrt{1 - \cos^2(2\pi\omega)}$$



$|\lambda| = 1$

$\det \phi = 1$

$\cos^2 2\pi\omega = 1$ $\qquad$ $2\omega = n$