Quasi-linear PDEs (I)

Method of characteristics and introduction to shock waves

First-order PDEs and useful notions

Consider a **first-order linear PDE**, in two variables, which can generally be expressed as:

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C_1(x,y)u = C_0(x,y)$$

This equation is called **homogeneous** if $C_0 \equiv 0$.

More generally, functions A, B, C_1 may depend on u; in this case, the first-order PDE of the form:

$$\boxed{A(x,y,u)\frac{\partial u}{\partial x} + B(x,y,u)\frac{\partial u}{\partial y} = C(x,y,u)}$$

is called <u>**quasi-linear**</u> (in two variables).

<u>**Remark:**</u>

<u>Every linear PDE is also quasi-linear</u>, because we can set:

$$C(x,y,u) = C_0(x,y) - C_1(x,y)u.$$

A prototypical -physically significant- example:

Transport (advection) equation

$$u_t + \boxed{c} u_x = 0$$

$$u_t + \boxed{c(x,t)} u_x = 0$$

Linear & homogeneous PDEs

$$u_t + c(x,t) u_x = \boxed{h(x,t)}$$

Linear & inhomogeneous PDE

$$u_t + \boxed{c(u)} u_x = 0$$

Quasi-linear PDE

Remark:

$$\left.\begin{array}{l} u_t + c(u)u_x = 0 \Rightarrow c'(u)u_t + c(u)c'(u)u_x = 0 \\ c'(u)u_t = c_t, \qquad c(u)c'(u)u_x = cc_x \end{array}\right\} \Rightarrow$$

$$c_t + cc_x = 0$$

Hopf (Riemann / inviscid Burgers) equation

Method of characteristics

Consider the Cauchy problem for the **quasi-linear PDE:**

$$\boxed{a(x,t,u)}\,u_t + \boxed{b(x,t,u)}\,u_x = c(x,t,u), \qquad u(x,0) = f(x)$$

- Introduce a curve **Γ** defined as: $\Gamma : \begin{cases} x = x(r), & x(0) = s \\ t = t(r), & t(0) = 0 \end{cases}$

- Assume that, on **Γ**: $u(x,t) = u(x(r), t(r))$, and differentiate wrt r:

$$\frac{du}{dr} = \frac{\partial u}{\partial t}\boxed{\frac{dt}{dr}} + \frac{\partial u}{\partial x}\boxed{\frac{dx}{dr}}$$

Next, require: $\dfrac{dt}{dr} = a, \quad \dfrac{dx}{dr} = b$

- Then, on **Γ**, the quasi-linear PDE is reduced to the ODE: $\boxed{\dfrac{du}{dr} = c}$

- The **equations:** $\boxed{\dfrac{dt}{dr} = a, \quad \dfrac{dx}{dr} = b, \quad \dfrac{du}{dr} = c} \Rightarrow \boxed{\dfrac{dt}{a} = \dfrac{dx}{b} = \dfrac{du}{c}}$

are called <u>the ***characteristics*** of the quasi-linear PDE.</u>

Lagrange-Charpit equations

Geometrical interpretation

Consider again the Cauchy problem for the **quasi-linear PDE:**

$$a(x,t,u)\, u_t + b(x,t,u)\, u_x = c(x,t,u), \qquad u(x,0) = f(x)$$

- Let $u(x,t)$ a **solution** of the PDE, and its **graph** $z = u(x,t)$, which is a **surface** in the **xtu-space.** The initial data which define a **curve γ** in the xt-plane (e.g., if $u(x,0)=f(x)$ then γ is the x-axis, etc) provides a **space curve Γ** that **lies on the graph**.

- Let $\mathbf{F}=(a, b, c)$ the vector field defined by PDE's coefficients
- The normal vector $\mathbf{N}$ to the surface z=u(x,t) is: $\mathbf{N}=(u_x, u_t, -1)$
- If $u(x,t)$ a **solution** of the PDE then

$$\boxed{\mathbf{F}\cdot\mathbf{N} = au_t + bu_x - c = 0 \Leftrightarrow \mathbf{F} \perp \mathbf{N} \Leftrightarrow \mathbf{F} \text{ tangent to } z = u(x,t)}$$

➧ **The graph of the solution can be constructed by finding the stream lines of $\mathbf{F}$ that pass through the initial curve Γ.**

1. Transport equation with const. velocity

- Simplest linear 1st-order problem: transport (advection) equation

$$u_t + c\,u_x = 0, \qquad u(x,0) = F(x)$$

$\omega = ck$, $\omega''(k) = 0$

Non-dispersive system

➢ Method of characteristics

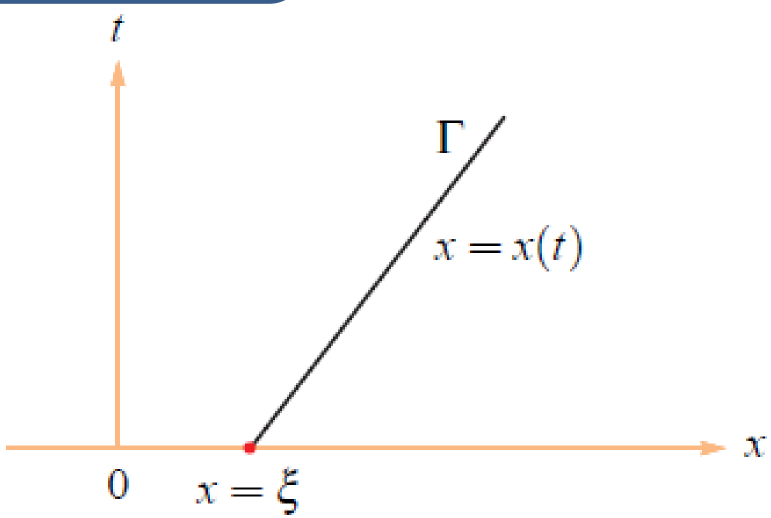
Reduce the problem to an ODE along some curve Γ: $x=x(t)$ such that $du/dt=0$

$$\frac{du(x(t),t)}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}\frac{dt}{dt} = \boxed{\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = 0} \qquad \frac{dx}{dt} = c$$

➡ $\dfrac{dx}{dt} = c \;\Rightarrow\; x(t) = c\,t + \xi$

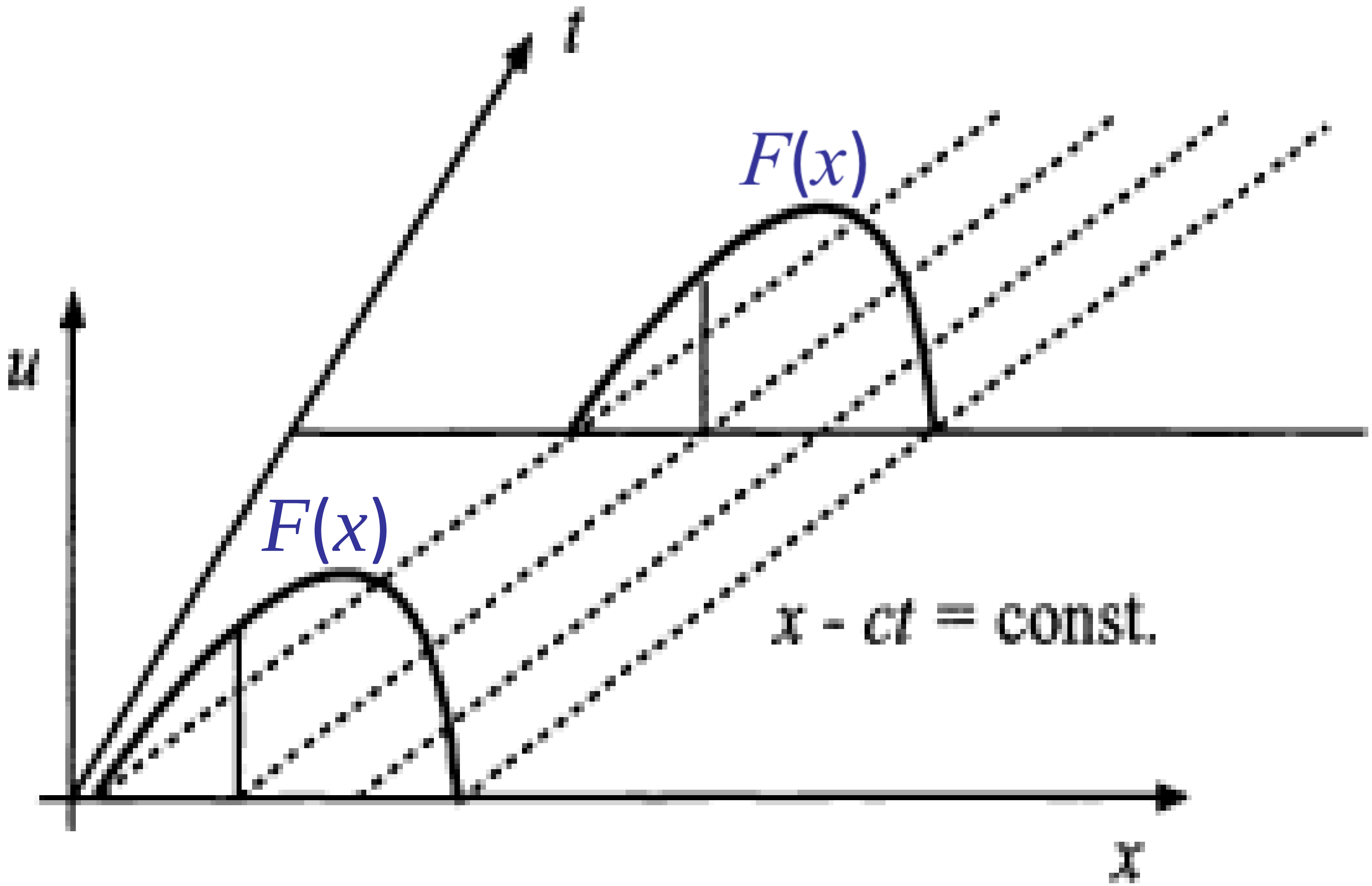
➡ $\dfrac{du}{dt} = 0 \;\Rightarrow\; u(x,t) = u(\xi,0) = F(\xi)$

General solution: $u(x,t) = F(x - c\,t)$



Solution of the transport equation

The IC, $F(x)$, is simply translated **without changing shape**



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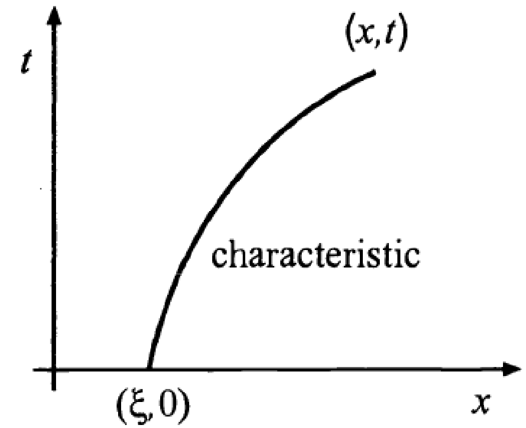
2. Transport equation with non const. velocity

Consider the Cauchy problem:

$$\boxed{\begin{aligned} &u_t + \boxed{2t}\, u_x = 0 \quad (c = 2t) \\ &u(x,0) = f(x) = \exp(-x^2) \end{aligned}}$$

On Γ: $x=x(t)$ we have $du/dt=0$:

$$\frac{du}{dt} = \frac{\partial u}{\partial t}\frac{dt}{dt} + \frac{\partial u}{\partial x}\frac{dx}{dt} = u_t + \boxed{\frac{dx}{dt}}\, u_x = 0$$



This leads to: $\dfrac{dx}{dt} = 2t, \quad x(0) = \xi \quad \Rightarrow \quad x(t) = t^2 + \xi$

Also, on Γ:

$$\left.\begin{aligned} &du/dt = 0 \Rightarrow u(x,t) = \text{const.} \\ &t = 0: \quad u(x,t) = u(\xi,0) \\ &\text{IC}: u(x,0) = f(x) \Rightarrow u(\xi,0) = f(\xi) \end{aligned}\right\} \Rightarrow \boxed{u(x,t) = f(\xi) = \exp[-(x - t^2)^2]}$$

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3. A boundary-value problem

Consider the BVP:
$$\begin{cases} \boxed{x^2} u_t + u_x + tu = 0, \quad x > 0, \; t \in \mathrm{R} \\ \text{BC at } x = 0: \quad u(0,t) = f(t) \end{cases}$$

On Γ: $t=t(x)$ we have: $$\frac{du}{dx} = \frac{\partial u}{\partial t}\frac{dt}{dx} + \frac{\partial u}{\partial x}\frac{dx}{dx} = \boxed{\frac{dt}{dx}} u_t + u_x$$

and we choose: $$\frac{dt}{dx} = x^2 \Rightarrow t = \frac{1}{3}x^3 + \tau$$ → our ξ in this case

On Γ: $t=t(x)$ we have: $$\boxed{\frac{du}{dx} = -tu} \Rightarrow \frac{du}{dx} = -\left(\frac{1}{3}x^3 + \tau\right)u$$ and thus:

$$\frac{du}{u} = -\left(\frac{1}{3}x^3 + \tau\right)dx \Rightarrow \int_{f(\tau)}^{u} \frac{du}{u} = \int_0^x -\left(\frac{1}{3}x^3 + \tau\right)dx \Rightarrow$$

$$\boxed{u = f(\tau)\exp\left(-\frac{1}{12}x^4 - \tau x\right) = f\left(t - \frac{1}{3}x^3\right)\exp\left(-\frac{1}{4}x^4 - xt\right)}$$

3. A nonlinear problem: Hopf equation

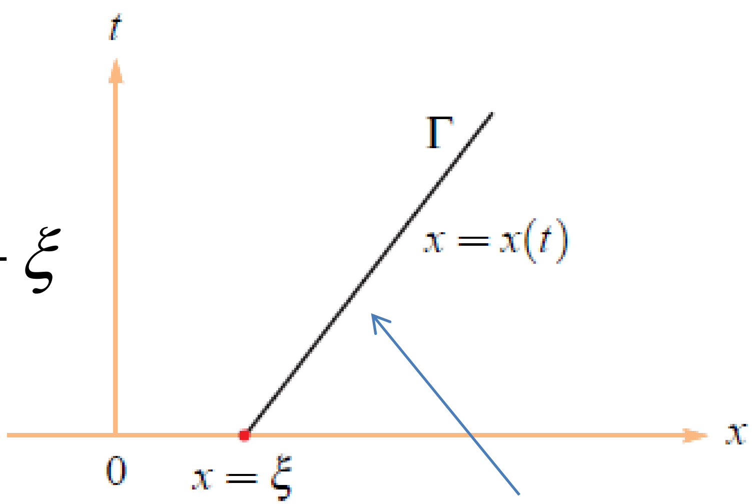
$$u_t + u\,u_x = 0, \quad u(x,0) = f(x), \quad x \in \mathrm{R},\ t > 0$$

On Γ: $x=x(t)$ we have: $du/dt=0$: $\dfrac{du}{dt} = \dfrac{\partial u}{\partial t}\dfrac{dt}{dt} + \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} = u_t + \dfrac{dx}{dt}u_x = 0$

As before:

$$du/dt = 0 \Rightarrow u(x,t) = \text{const.}$$

$$dx/dt = u, \quad x(0) = \xi \quad \Rightarrow \quad x(t) = ut + \xi$$



Characteristics are again straight lines

Since:

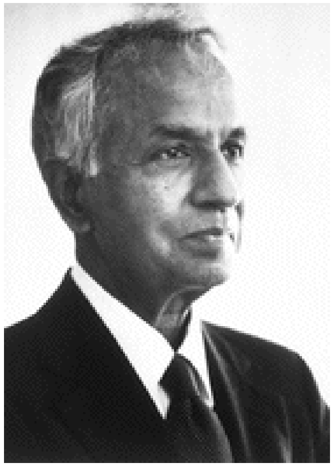
$$u(x,0) = f(x) \Rightarrow u(\xi,0) = f(\xi)$$

$$\left.\begin{aligned} u(x,t) = u(\xi,0) = f(\xi) \\ \xi = x - ut \end{aligned}\right\} \Rightarrow \boxed{u = f(x-ut)}$$

Implicit solution of the Hopf equation

An explicit solution of the Hopf equation

S. Chandrasekhar found (1943) an **explicit solution** of the following IVP:

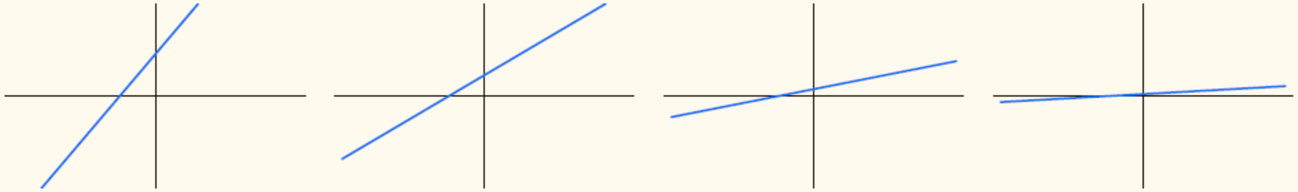


$$\boxed{u_t + u\,u_x = 0, \qquad u(x,0) = f(x) = ax + b}$$

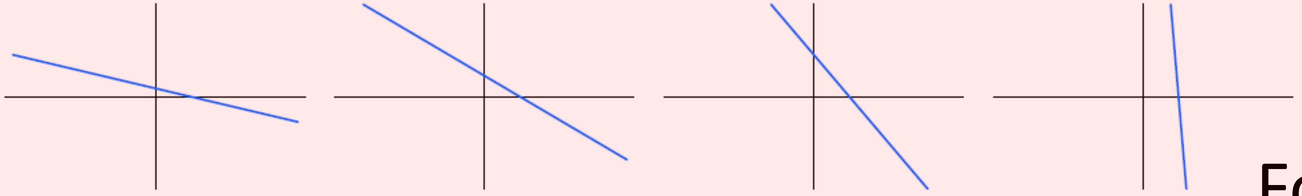
We have found: $\quad u = f(x - ut)$

and thus: $u = a(x - ut) + b \Rightarrow \boxed{u = \dfrac{ax+b}{1+at}}$

$a > 0$: the solution **flattens** as $t \to \infty$



Rarefaction wave

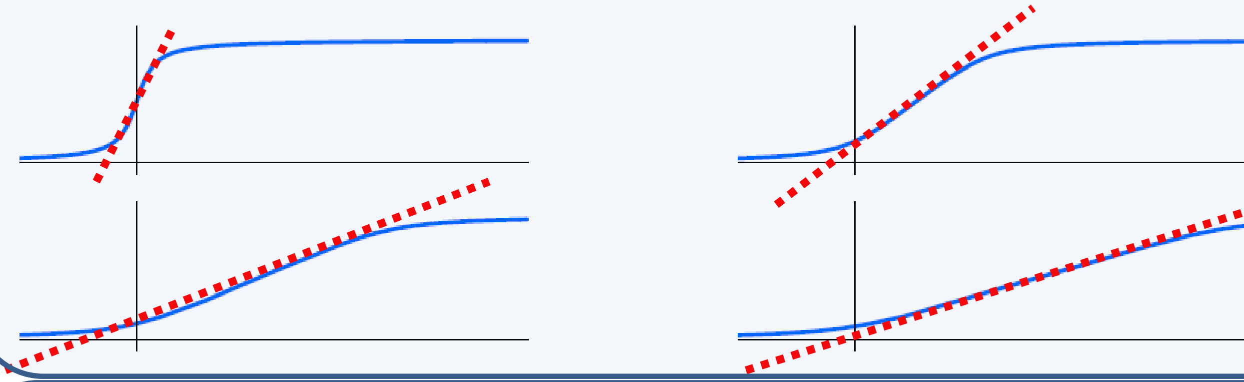


Shock wave

For $t = t_B = -1/a$ the solution **blows up**

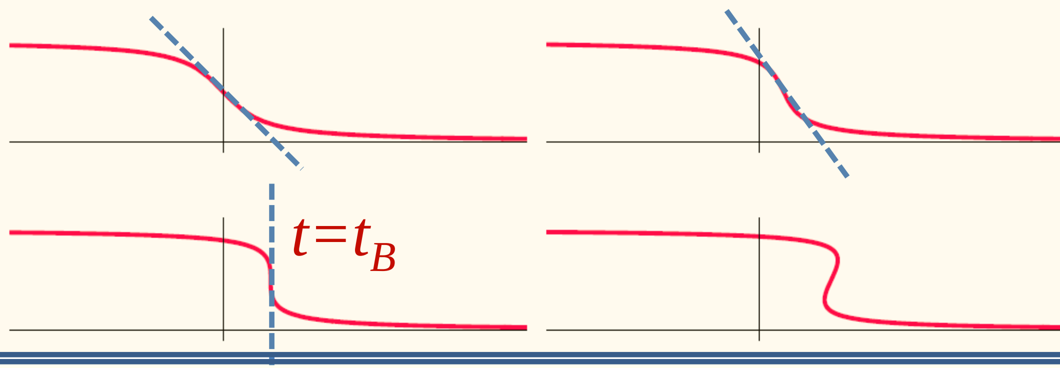
$a < 0$: the solution **steepens** as $t \to \infty$

What can we learn from the explicit solution?



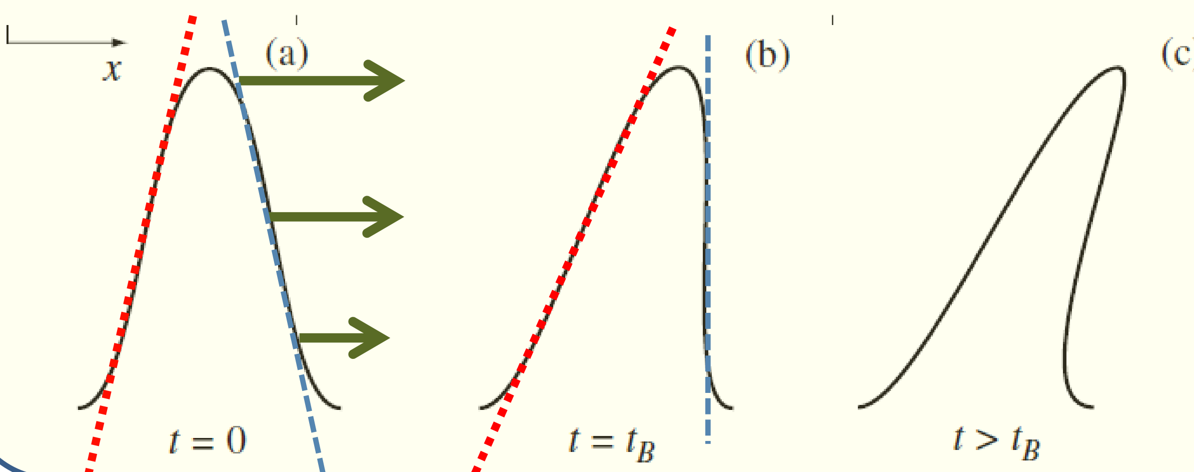
Positive slope

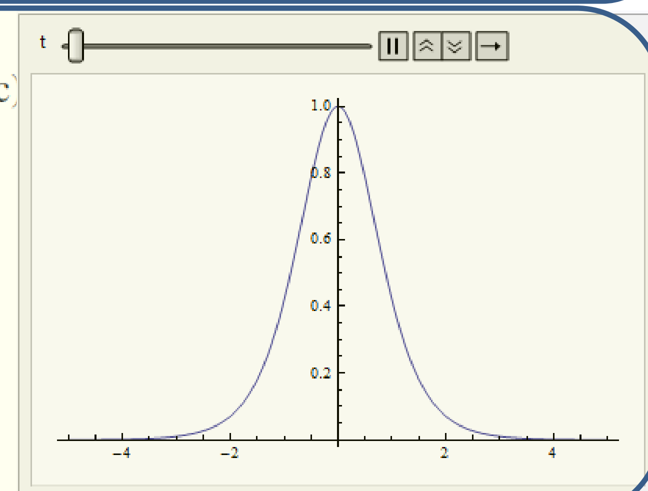
Rarefaction wave



Negative slope

Shock wave





Breaking time

Consider the general problem: $u_t + u\,u_x = 0,\ u(x,0) = f(x)$

We have found: $\left.\begin{array}{l} x = ut + \xi \\ u = f(\xi) \end{array}\right\} \Rightarrow \underline{x = \xi + f(\xi)t \quad (1)}$

We wish to determine the **breaking time t_B** occurring when the profile of the solution develops an infinite slope:

$$\left.\begin{array}{l} u_x = \dfrac{\partial u}{\partial x} = \dfrac{du}{d\xi}\dfrac{\partial \xi}{\partial x} = f'(\xi)\xi_x \\ (1),\ \partial_x:\ 1 = \xi_x + f'(\xi)t\,\xi_x \Rightarrow \xi_x = \dfrac{1}{1+f'(\xi)t} \end{array}\right\} \Rightarrow \boxed{u_x = \dfrac{f'(\xi)}{1+f'(\xi)t}}$$

If ***f* '(ξ) > 0** $\forall x$ then the solution is **finite** $\forall t \rightarrow$ rarefaction wave

If ***f* '(ξ) < 0** the solution breaks up at the **earliest critical time**:

$$\boxed{t_B = \min_{\xi>0}\{-1/f'(\xi) | f'(\xi) < 0\}}$$

What happens before the breaking time t_B

Using $\boxed{x = \xi + f(\xi)t \quad (1)}$ we found: $\boxed{u_x = \dfrac{f'(\xi)}{1+f'(\xi)t} \quad (2)}$

Similarly:

$$\left.\begin{aligned} & u_t = \frac{\partial u}{\partial t} = \frac{du}{d\xi}\frac{\partial \xi}{\partial t} = f'(\xi)\xi_t \\ & (1),\ \partial_t:\ 0 = \xi_t + f'(\xi)t\,\xi_t + f(\xi) \Rightarrow \xi_t = -\frac{f(\xi)}{1+f'(\xi)t} \end{aligned}\right\} \Rightarrow$$

$$\boxed{u_t = -\frac{f(\xi)f'(\xi)}{1+f'(\xi)t} \quad (3)}$$

Hence, from (2)-(3), and for $\boldsymbol{t < t_B}$, the solution remains **single-valued** and satisfies the Hopf equation:

$$u_t + uu_x = -\frac{f(\xi)f'(\xi)}{1+f'(\xi)t} + f(\xi)\frac{f'(\xi)}{1+f'(\xi)t} = 0 \quad \checkmark$$

What happens for times $t \geq t_B$: Characteristics intersect

The slope of a characteristic passing (x^0, t^0) is: $\boxed{\dfrac{dx}{dt} = c(f(x^0))}$

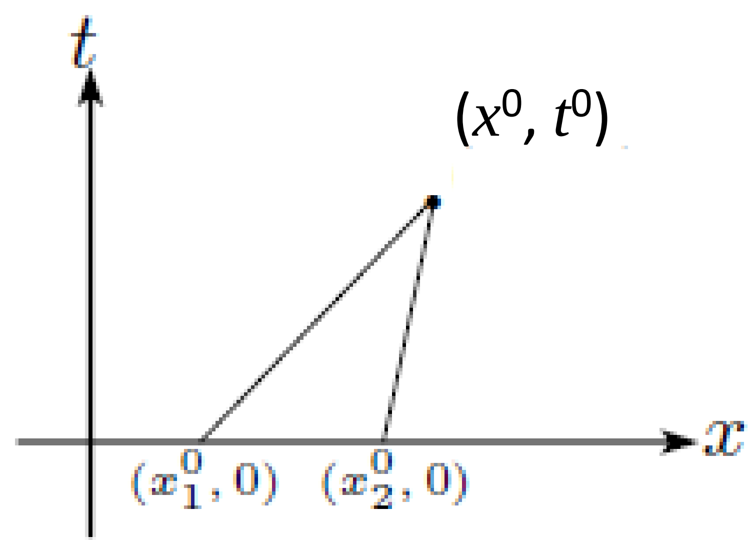
Here, $c = c(u)$ is a **strictly increasing** function:

$$u_t + c(u)u_x = 0,\ \ c(u) = u \Rightarrow c'(u) = 1 > 0$$

Let f be a **strictly decreasing** function; then, $c(f)$ is also **strictly decreasing:**

$$x_1^0 < x_2^0 \rightarrow c(f(x_1^0)) > c(f(x_2^0))$$

and, hence, characteristics passing through $(x_1^0,0),\ (x_2^0,0)$ **intersect!**

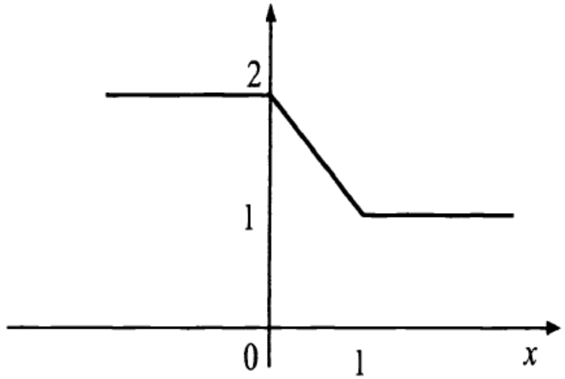


At the intersection point, the solution $u(x,t)$ becomes **multi-valued** because it takes both values $f(x_1^0),\ f(x_2^0)$

An example

Consider the IVP: $u_t + u u_x = 0, \quad u(x,0) = f(x), \quad x \in \mathrm{R}$

$$f(x) = \begin{cases} 2, & x < 0 \\ 2 - x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$



We have found: $\left.\begin{aligned} x &= ut + \xi \\ u &= f(\xi) \end{aligned}\right\} \Rightarrow x = \xi + f(\xi)t$

Characteristics:

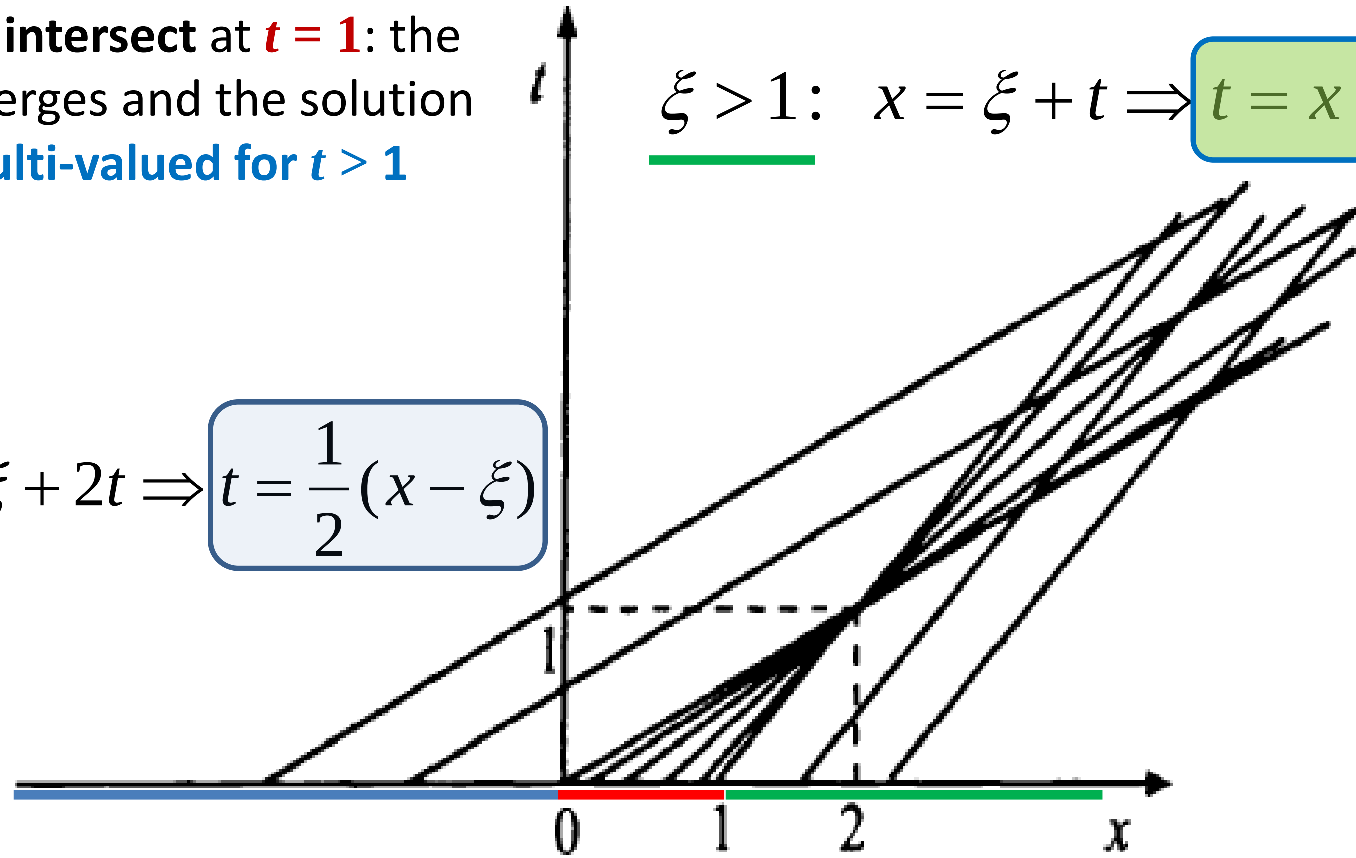
$$f(\xi) = \begin{cases} 2, & \xi < 0 \\ 2 - \xi, & 0 \le \xi \le 1 \\ 1, & \xi > 1 \end{cases}$$

$$\boxed{x = \begin{cases} \xi + 2t, & \xi < 0 \\ \xi + (2 - \xi)t, & 0 \le \xi \le 1 \\ \xi + t, & \xi > 1 \end{cases}}$$

An example (cont.) - characteristics

Characteristics intersect at $t = 1$: the **shock wave** emerges and the solution becomes **multi-valued for $t > 1$**

$$\xi > 1: \quad x = \xi + t \Rightarrow \boxed{t = x - \xi}$$



$$\xi < 0: \quad x = \xi + 2t \Rightarrow \boxed{t = \frac{1}{2}(x - \xi)}$$

$$0 \le \xi \le 1: \quad x = \xi + (2 - \xi)t \Rightarrow \boxed{t = \frac{x - \xi}{2 - \xi}}$$

An example (cont.) – breaking time

How to determine the **breaking time t_B**:

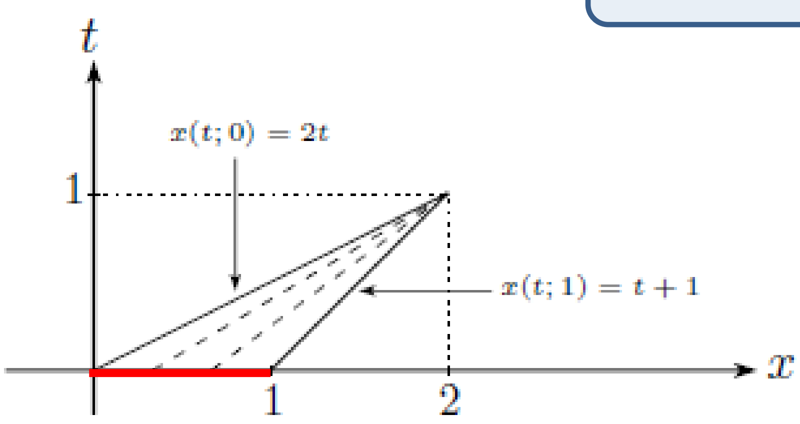
Recall that: $\boxed{t_B = \min_\xi\{-1/f'(\xi) | f'(\xi) < 0\}}$

Here, we have: $f(\xi) = \begin{cases} 2, & \xi < 0 \\ 2-\xi, & 0 \le \xi \le 1 \\ 1, & \xi > 1 \end{cases}$ and hence $\boxed{t_B = 1}$

Alternatively, recall that we found: $x = \xi + (2-\xi)t, \quad 0 \le \xi \le 1$

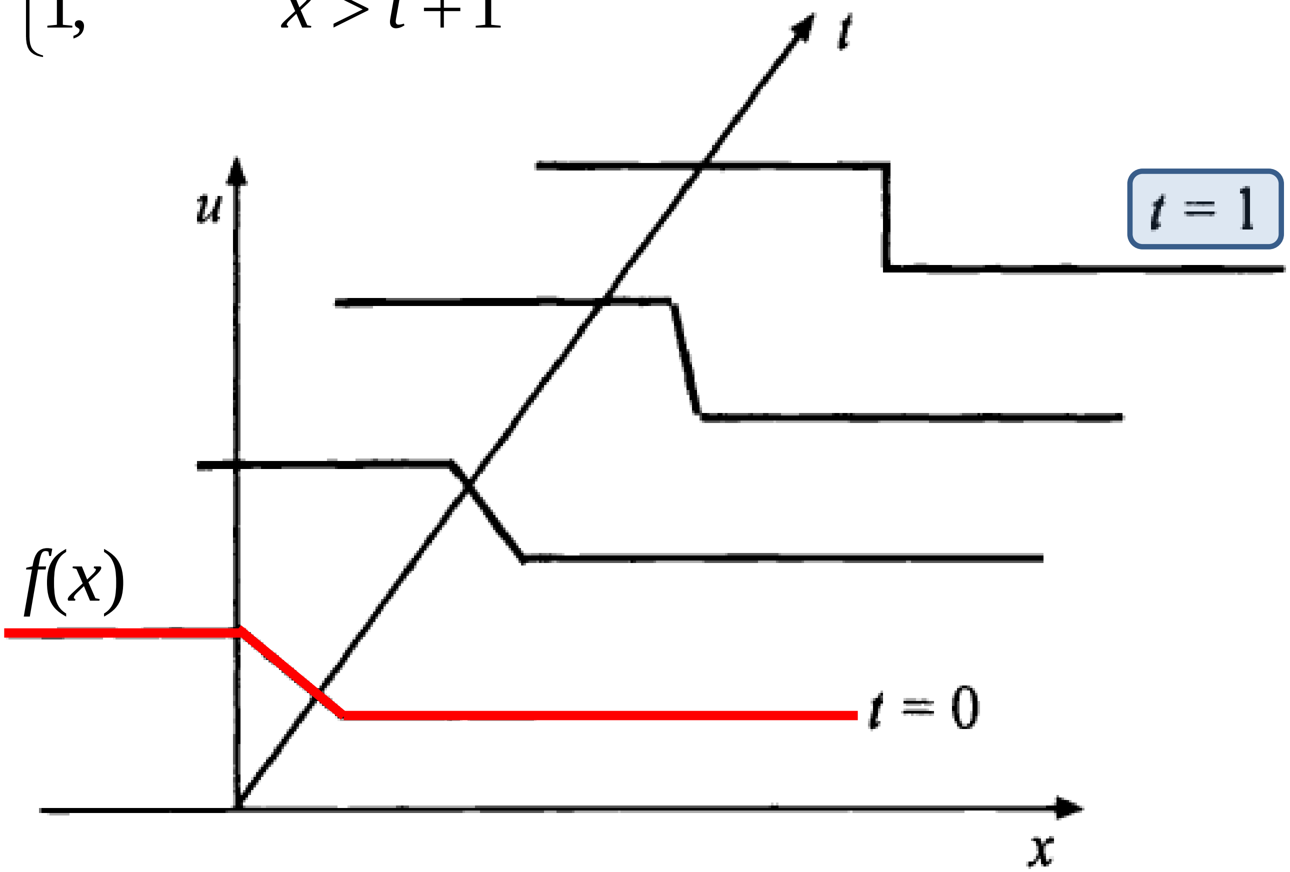
and so $\left.\begin{array}{l} \underline{\xi = 0:}\ \ x = 2t \\ \underline{\xi = 1:}\ \ x = 1+t \end{array}\right\} \Rightarrow$characteristics intersect at $\boxed{(x,t)=(2,1)}$

All other characteristics in this interval pass $(x,t) = (2,1)$



An example (cont.) – shock formation

Solution: $$u(x,t)=\begin{cases}2, & x<2t\\ \dfrac{2-x}{1-t}, & 2t\le x\le t+1\\ 1, & x>t+1\end{cases}$$



Exercises – linear problems

1) Use the method of characteristics to solve the IVPs:

1.1) $$u_t + u_x + u = 0, \quad u(x,0) = f(x) = \cos x, \quad x \in \mathrm{R}$$

1.2) $$u_t + 2xt\, u_x = u, \quad u(x,0) = f(x) = x, \quad x \in \mathrm{R}$$

2) Use the method of characteristics to show that the solution of the IVP:

$$u_t + c u_x = h(x,t), \quad u(x,0) = f(x), \quad x \in \mathrm{R}$$

is: $$u(x,t) = f(x - ct) + \int_0^t h(x - c(t - t'), t')\, dt'$$

In all cases confirm, by direct substitution, that the solution you found satisfies the corresponding PDE.

Exercises – nonlinear problems

3) Use the method of characteristics to solve the Hopf equation:

$$u_t + uu_x = 0, \quad u(x,0) = f(x)$$

in the following cases:

3.1) $$f(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

3.2) $$f(x) = \begin{cases} 0, & x < 0 \\ -x, & 0 \le x \le 1 \\ -1, & x > 1 \end{cases}$$

In both cases:

a) Draw the characteristics in the xt-plane

b) Write down the solution and draw some characteristic snapshots of the solution at different time instants

c) Determine the breaking time (when relevant)