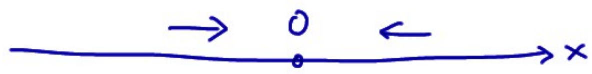
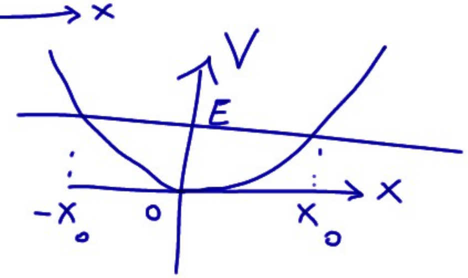
2ο θέμα 9/5/2019

$$F = -kx^3$$




$$V = -\int F\,dx = \frac{kx^4}{4} + \cancel{c}^{\,0}$$




(α) Περίοδος διαστατικά

$$T \propto m^{\alpha} k^{\beta} x_0^{\gamma} \Rightarrow \ldots$$

δες λύση στο eclass

$F \quad x_0 \quad m$

$$F = m\alpha = m\frac{x_0}{T^2}$$

$$\Rightarrow T^2 = \frac{m x_0}{F}$$

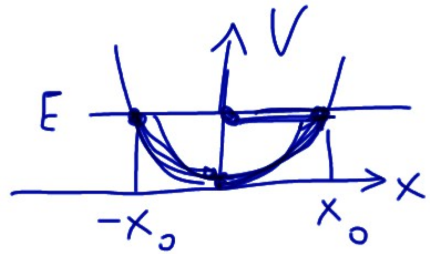
$$T \propto \sqrt{\frac{m x_0}{k x_0^3}} \propto \sqrt{\frac{m}{k x_0^2}}$$

(Αν είχα $F = -k\,x^{2n+1}$ τότε

$$F = m\frac{x}{t^2} \rightarrow t = \sqrt{\frac{m x}{F}}$$ δηλ. $$T \propto \sqrt{\frac{m x_0}{k x_0^{2n+1}}} = \sqrt{\frac{m}{k x_0^{2n}}}$$

Μόνο για τον αρμονικό ταλαντωτή $(n=0)$, η $T$ ανεξάρτητη του $x_0$)

(β) $$T = 4\int_0^{x_0} \frac{dx}{\dot{x}}$$




$$\frac{m\dot{x}^2}{2} + \frac{k x^4}{4} = E = \frac{k x_0^4}{4}$$

$E = V(x)$ για $x = \pm x_0$

$$\dot{x} = + \sqrt{\frac{2k}{4m}\left(x_0^4 - x^4\right)}$$

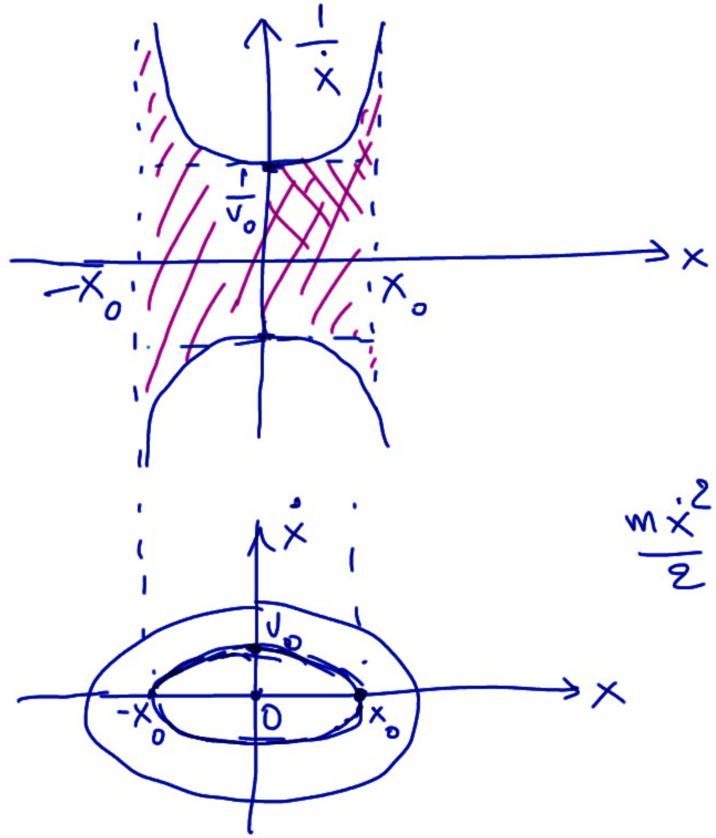
και άρα $$T = 4\int_0^{x_0} \frac{dx}{\sqrt{\frac{2k}{4m}\left(x_0^4 - x^4\right)}}$$

$$\int_0^1 \frac{d\xi}{\sqrt{1-\xi^4}} = 1.311$$

Με $x = x_0 \xi$, $T = \dots\dots = 7.416\sqrt{\frac{m}{k x_0^2}}$

$$\int \frac{dx}{\dot{x}} = \int \frac{1}{\dot{x}}\, dx$$

$$T > 4\, x_0 \frac{1}{v_0}$$




$$\frac{m\dot{x}^2}{2} + \frac{kx^4}{4} = E$$

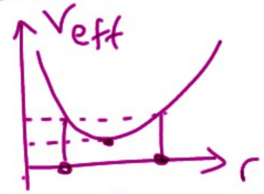
Αν έχω διδιάστατη κίνηση $\bar{F} = -k r^3 \hat{r}$

τι ταχύτητα έπρεπε να δώσω στο σώμα για να κάνει κυκλική τροχιά ακτίνας $r_0$;


Λύση:

$$|F| = \frac{mv^2}{r} \Leftrightarrow k r^3 = \frac{mv^2}{r} \Leftrightarrow v = \sqrt{\frac{k}{m}}\, r^2$$

Αν διαταράξω αυτή την τροχιά ποια η περίοδος των ακτινικών ταλαντώσεων;




Λύση:

$$V = -\int \bar{F}\cdot d\bar{r} = \frac{k r^4}{4}, \quad V_{eff} = \frac{L^2}{2mr^2} + \frac{k r^4}{4}$$

$$\frac{m\dot{r}^2}{2} + V_{eff}(r) = E \qquad r = r_0 + q, \quad V_{eff}(r) \approx V_{eff}(r_0) + \cancel{V'_{eff}(r_0)\, q} + \frac{1}{2} V''_{eff}(r_0)\, q^2$$

$$\frac{m}{2}\dot{q}^2 + \frac{1}{2} V''_{eff}(r_0)\, q^2 = \sigma\tau\alpha\vartheta \xrightarrow{\text{παραγωγίζω}} \ddot{q} + \frac{V''_{eff}(r_0)}{m}\, q = 0 \rightarrow \omega^2, \quad T = 2\pi/\omega$$

Αν θέλω την διαταραγμένη τροχιά:

$$\frac{m\dot r^2}{2} + V_{eff}(r) = E \xrightarrow{\dot r = \frac{dr}{d\varphi}\dot\varphi = \frac{L}{mr^2}\frac{dr}{d\varphi}} \frac{m}{2}\frac{L^2}{m^2 r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{L^2}{2mr^2} + \frac{kr^4}{4} = E$$

$$u'' + u = -\frac{mF}{L^2u^2} = -\frac{m(-k/u^3)}{L^2u^2} \Leftrightarrow u'' = f(u)$$

με $f(u) = -u + \frac{mk}{L^2u^5}$ , $f'(u) = -1 - \frac{5mk}{L^2u^6}$

Θέτω $u = \frac{1}{r_0} + q$ και έχω $q'' = f\left(\frac{1}{r_0}+q\right) = \cancelto{0}{f\left(\frac{1}{r_0}\right)} + f'\left(\frac{1}{r_0}\right) q$

$$q'' + \underbrace{\left(1+\frac{5mk}{L^2}r_0^6\right)}_{\lambda^2} q = 0 \qquad q = q_0\cos(\lambda\varphi + C)$$

$$u = \frac{1}{r_0} + q_0\cos(\lambda\varphi + C)$$


$$r = \frac{1}{\frac{1}{r_0} + q_0\cos(\lambda\varphi + c)} = \frac{r_0}{1 + q_0 r_0\cos(\lambda\varphi + c)}$$

$\Delta\varphi = 2\pi/\lambda - 2\pi$

$r_{min} = \frac{r_0}{1+q_0r_0}$

$r_{max} = \frac{r_0}{1-q_0r_0}$

$$f(u) = -u + \frac{mk}{L^2 u^5} \qquad u = \frac{1}{r_0} + q$$

$$\underbrace{f(u)}_{f\left(\frac{1}{r_0}+q\right)} \approx f\left(\frac{1}{r_0}\right) + f'\left(\frac{1}{r_0}\right)q$$

$$f(x) \overset{x - x_0 = q}{=\!=\!=\!=} f(x_0 + q) = f(x_0) + f'(x_0)\,q + \frac{1}{2} f''(x_0)\,q^2 + \dots$$

$$m\left(\ddot{r} - r\dot{\varphi}^2\right) = F \iff m\ddot{r} = F + \frac{L^2}{mr^3} = F_{eff}(r)$$

$$\frac{1}{r}\frac{d}{dt}\left(mr^2\dot{\varphi}\right) = 0 \longrightarrow \dot{\varphi} = \frac{L}{mr^2} \qquad -\frac{dV_{eff}}{dr}$$

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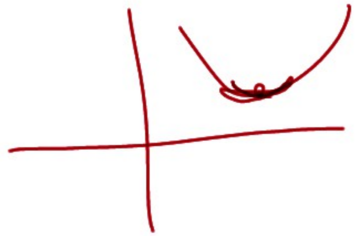
$$m\ddot{x} = F(x)$$



$$x = x_0 + q$$

$$m\ddot{q} = \cancel{F(x_0)} + F'(x_0)\,q$$

$$\ddot{q} \left(-\frac{F'(x_0)}{m}\right) q = 0 \quad \rightarrow \omega^2$$

$$\left(\text{ίδιο με} \quad \omega^2 = \frac{V''(x_0)}{m}\right)$$

$$V \approx V(x_0) + \cancel{V'(x_0)\,q} + \frac{1}{2}V''(x_0)\,q^2$$