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ARTICLES

What Kinds of Numbers Do Students Assign to Literal Symbols? Aspects of the Transition from Arithmetic to Algebra Q1

Konstantinos P. Christou and Stella Vosniadou
Department of Philosophy and History of Science, National and Kapodistrian University of Athens

Three experiments used multiple methods—open-ended assessments, multiple-choice questionnaires, and interviews—to investigate the hypothesis that the development of students' understanding of the concept of real variable in algebra may be influenced in fundamental ways by their initial concept of number, which seems to be organized around the notion of natural number. In the first two experiments 91 secondary school students (ranging in age from 12.5 to 14.5 years) were asked to indicate numbers that could or could not be used to substitute literal symbols in algebraic expressions. The results showed that there was a strong tendency on the part of the students to interpret literal symbols to stand for natural numbers and a related tendency to consider the phenomenal sign of the algebraic expressions as their "real" sign. Similar findings were obtained in a third, individual interview study, conducted with tenth grade students. The results were interpreted to support the interpretation that there is a systematic natural number bias on students' substitutions of literal symbols in algebra.

INTRODUCTION

The experiments presented in this article investigate the hypothesis that there is a strong tendency in students to substitute literal symbols in algebra only with natural numbers. Prior research has

[1]By "variable" we mean "real variable," that is a variable whose range is a subset of the real numbers. Also, the term "concept" is used to refer to the normative, socially shared, and culturally accepted concept, while the term "conception" is used in this paper to refer to individuals' subjective constructions of the mathematical concept (see also Leinhardt, Zaslavsky, & Stein, 1990; Sfard, 1991). The term "initial concept" is also used to refer to what is assumed to be the initial, psychological, concept of number, which we argue is close to the mathematical notion of natural number.

Correspondence should be sent to Stella Vosniadou, Department of Philosophy and History of Science, National and Kapodistrian University of Athens. E-mail: svosniad@phs.uoa.gr

concentrated on students' difficulties in understanding that literal symbols in algebra stand for more than one number. We argue, however, that the development of students' understanding of the concept of real variable is also influenced by their initial concept of number, which seems to be organized around the notion of natural number.[1] In other words, there is a natural number bias that influences students' substitutions of literal symbols in algebra causing further difficulties, even after the notion of generalized number has been achieved. To the best of our knowledge this tendency has not been investigated systematically so far, although acknowledged by some researchers (Booth, 1984; Graham & Thomas, 2000; Kuchemann, 1981; Malara & Iaderosa, 2000; Schmittau, 2005). In the pages that follow we start with a selective review of the literature on students' use of literal symbols in algebra and continue to develop our theoretical framework and hypotheses.

Selective Review of Prior Research

The use of literal symbols is an important source of difficulty in students' transition from arithmetic to algebra (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Hart, 1981; Nathan & Koellner, 2007). Many students refer explicitly to the use of literal symbols as their main problem with algebra, saying that they understood mathematics until literal symbols appeared (Sackur, 1995). Prior research has shown that some students have no idea about how to interpret such literal symbols, thinking that they stand for abbreviated names of people or objects or as coded numbers, with values corresponding to their positions in the alphabet (Booth, 1984, 1988; Kuchemann, 1981; Wagner, 1981). While such mistakes are usually abandoned by tenth grade (Asquith, Stephens, Knuth, & Alibali, 2007; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Stacey & MacGregor, 1997), some other difficulties are more persistent and not so easily corrected. One such difficulty is the tendency to think that literal symbols can stand for only one, specific, unknown number as opposed to the generalized number (Asquith et al., 2007; Booth, 1988; Collis, 1975; Knuth et al., 2005; Kuchemann, 1981). Booth (1984) found that such difficulties persist even when students receive detailed instructions to correct them.

Students' difficulties with the generalized number were originally explained in terms of Piagetian theory. More specifically, it was claimed that generalized number is an abstract concept that cannot be understood until students reach the stage of formal operations (Collis, 1975; Kuchemann, 1978, 1981). This interpretation was originally criticized on the grounds that there is no empirical evidence to support it (Booth, 1988; Stacey & MacGregor, 1997). Recent research has shown, however, that students can deal with abstract mathematical entities, such as the generalized number, much earlier than the Piagetian theory would predict (Carraher, Schliemann, & Brizuela, 2001; Schmittau, 2005).

Another explanation focuses on the negative role that can be possibly played by prior knowledge in arithmetic (Booth, 1984; Herscovics & Linchevski, 1994; Kieran, 1992; MacGregor & Stacey, 1997; Matz, 1980; Stacey & MacGregor, 1997). For example, Booth has argued that students carry in algebra their erroneous knowledge of arithmetic, while Matz has claimed that students' difficulties reflect inappropriate use of arithmetic to interpret the use of literal symbols in algebra.

In general, many researchers agree that arithmetic is a system that is procedural and concrete and where transformations take place in specific numbers resulting in numerical responses, while algebra is more structural and abstract and where transformations take place with/on algebraic expressions and not with/on specific numbers (Booth, 1984; Graham & Thomas, 2000; Kieran, 1992; Lee & Wheeler, 1981; Linchevski & Herscovics, 1996; Nickson, 2000; Schmittau, 2005). In other words, there is a fundamental difference between the arithmetic and algebraic system of conventions—often referred to as the "cognitive gap" (Herscovics & Linchevski, 1994), "cut point," or "didactic cut" (Filloy & Rojano, 1989)—which can account for a series of mistakes and difficulties in students' learning of algebra, such as the "lack of closure" error, which refers to students' unwillingness to accept algebraic expressions containing literal symbols as final answers (Collis, 1975; Sfard & Linchevski, 1994; Tirosh, Even, & Robinson, 1998). Despite the noted differences between arithmetic and algebra, there are a number of possible ways to use students' knowledge of arithmetic to facilitate learning in algebra (e.g., Carpenter & Fennema, 1992; Carraher et al., 2001).

When it comes to explain why it is so difficult for students to understand literal symbols as generalized number, some researchers refer to the duality of mathematical ideas (Sfard, 1991). Sfard has argued that mathematical ideas can be viewed both as a process and as a product of this process, a product that can also stand on its own right as a self-existent entity. Other researchers have proposed similar models such as the process/concept duality (Gray & Tall, 1994), the process/structure distinction (Kieran, 1992), and the APOS theory, which is an acronym of the sequence: action, process, object, and schema (Dubinsky, 1991). Using this distinction, algebraic expressions can be conceived both as the process of the calculations notated and also as algebraic entities on their own right.

According to these approaches, students initially understand mathematical ideas procedurally. A more sophisticated understanding develops when students understand the structural side of mathematical ideas and even more as they become able to switch flexibly between the procedural and the structural views. This is what Sfard (1991) characterized as reification. According to this position, only after practice on the procedural level is it possible to develop a higher level of understanding, that is, the structural aspect of a mathematical idea (see also Kieran, 1992). Arriving at a new structural description of the number concept, Sfard (1991) argued "is an ontological shift, a qualitative jump. Such conceptual upheaval is always a rather complex phenomenon, especially when it is accompanied by subtle alterations of meaning and applications" (p. 30).

We agree with Sfard's position that it is useful to analyze students' difficulties with the number concept as requiring an ontological shift. From our perspective, this ontological shift complicates the transition from arithmetic to algebra. More specifically, we argue that there is a "natural number bias" that influences students' substitutions of literal symbols even when the notion of generalized number has been understood. This position will be described in the next section.

Students' Conceptions of Number and the Interpretation of Literal Symbols in Algebra

The kind of numbers students think literal symbols can represent is an important source of meaning for the use of literal symbols in algebra. Most researchers who study the development of students' mathematical ideas agree that algebraic expressions can take their meaning not only from the formal system of algebra but also from their reference, that is the quantities and relations between quantities to which they refer, as well as from their linkage to the "world of numbers" (see Kieran, 2007; Resnick, 1987). Even when students accept the more abstract, structural aspect of an algebraic expression as an entity on its own right, they do not necessarily disengage this entity from its initial reference to numbers and relations between numbers (Arcavi, 1994; Sfard, 1995). The reference of literal symbols—the numbers literal symbols stand for—is an important source of meaning, which influences performance in various mathematical tasks (Crowley, Thomas, & Tall, 1994; Demby, 1997; Graham & Thomas, 2000; Kaput, 1989; Kieran, 1992; Malara, 1999; Warren & Pierce, 2004). For instance, "guess and check" with specific numbers has often proven to be an effective strategy for dealing with mathematical problems (see Cerulli & Mariotti, 2001; Johanning, 2004) despite the fact that it rarely provides mathematically accepted solutions. Moreover, the reference to specific numbers can result in difficulties in a range of other mathematical tasks, depending on the kinds of numbers used. It has been amply demonstrated, for example, that the kind of multiplicand used has an impact on problem solution when the multiplier is smaller than one (Fischbein, Deri, Nello, & Marino, 1985; Harel, Behr, Post, & Lesh, 1994).

It would be reasonable to assume that students will find it easy to understand that literal symbols stand for any real number, that is, natural or non-natural, rational or irrational, positive or negative, because they usually have years of instruction at least on rational number before they are introduced to the concept of variable. However, research shows that even at the end of high school many students do not have a unified concept of number as real number, and that their ideas remain tied around natural numbers and their properties (Moss, 2005; Vamvakoussi & Vosniadou, 2007).

Students' number knowledge is based on their initial experience with counting numbers (see Gelman, 2000). Such a conceptualization of number is already apparent in the preschool years and is further confirmed and strengthened during the first years of primary school, when students learn systematically the symbolism and the properties of natural numbers (Vosniadou, Vamvakoussi, & Skopeliti, 2008). It is also strengthened by the continuous use of natural numbers throughout the mathematics curriculum. Although there are positive ways to build on students' knowledge of natural numbers to introduce them to non-natural numbers (Mack, 1988; Olive, 1999; Steffe, 1992), these are not usually taken into consideration in instruction. The result is that students have persistent difficulties and misconceptions caused by the interference of natural number on their interpretation of rational and real numbers (Behr, Harel, Post, & Lesh, 1992; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2007; Verschaffel & Vosniadou, 2004). This interference is known in the literature as the "whole number bias" (Ni & Zhou, 2005).

Vosniadou and her colleagues (Vosniadou, 2003; Vosniadou et al., 2008) have argued that the initial understanding of number, constructed around the notion of natural number, constitutes a "framework theory" that guides the development of more advanced mathematical ideas. The term "framework theory" is used to denote a skeletal structure, which is however generative in the sense that it can generate explanations and predictions, and not an explicit, well-formed, and socially shared scientific theory (Vosniadou et al., 2008). It is possible that many components of the initial framework theory of number are based on children's experiences, an assumption consistent with the " knowledge-in-pieces" approach (diSessa, 2008; Schoenfeld, Smith, & Arcavi, 1993). According to diSessa (2008), the physical knowledge system of novices consists of an unstructured collection of many, relatively isolated, self-explanatory elements known as phenomenological primitives (p-prims) that originate from superficial interpretations of physical reality. Learning requires collecting and systematizing those pieces of knowledge into larger systems of more complex knowledge structures, such as physics laws.

Although we agree that such phenomenological primitives may form part of the knowledge system of novices, we nevertheless believe that these elements are integrated from early on, under the influence of lay culture and language, into a skeletal conceptual structure, which guides the knowledge acquisition process. Within this larger conceptual structure, or framework theory, number is essentially natural number. Students' initial framework theory of number encompasses a number of interrelated beliefs that underlie students' expectations about what counts as number and how it is supposed to behave—namely, that numbers have a distinct symbolic representation, refer to absolute quantities, have a distinct successor and obey the principal of discreteness, and that they behave like natural numbers with respect to ordering and operations (Vamvakoussi & Vosniadou, 2010).

The framework theory approach is a constructivist approach and as such it assumes that learning is built on prior knowledge mainly through additive, enrichment types of mechanisms, such as assimilation, internalization, and appropriation. In the great majority of cases these mechanisms promote learning. Sometimes, however, the incoming information is incompatible with what is already known, requiring the revision of existing ideas. In such cases, the use of additive learning mechanisms can destroy the coherence of the initial conceptual structure creating either fragmentation (diSessa, 2008) or systematic misconceptions—otherwise known as synthetic conceptions (Vosniadou et al., 2008). For example, many students believe that multiplication always makes a number bigger (Fischbein et al., 1985), that longer decimals are bigger (Moss, 2005), that the bigger the numerator and the denominator the bigger the fraction (Mack, 1988; Stafylidou & Vosniadou, 2004), and that discreteness may apply to decimals but not to fractions (Vamvakoussi & Vosniadou, 2010). Thus, although students' exposure to instruction and texts usually triggers top-down coherence in their knowledge organization (diSessa, 1993), we believe that fragmentation, or synthetic conceptions, can be a first product of instruction in cases when a rather coherent but incompatible structure is already established.

If we are correct in that there is such an initial understanding of number that influences the knowledge acquisition process in fundamental ways, it makes sense to assume that it would also influence the way students interpret the use of literal symbols. For example, it could be hypothesized that students may start by thinking that literal symbols refer only to natural numbers, and continue to think this way even after they have understood the notion of generalized number. In addition, the framework theory approach would predict that the process of understanding literal symbols as standing for any real number would not be an all or nothing process, but would be characterized by intermediate steps. Those steps would have the characteristics either of internally inconsistent (fragmented) conceptions (diSessa, 1993) or of synthetic conceptions (Vosniadou et al., 2008), reflecting students' attempts to make sense of the new information to which they are instructed. In other words, some students may be inconsistent in their substitutions, sometimes thinking that fractions, for example, can be substituted for literal symbols and some other times not. Alternatively, they may form a synthetic conception according to which, for instance, only decimals but not fractions can be substituted for literal symbols in a systematic way.

Interestingly, there is some evidence coming from the history of mathematics that seems to support this point of view. For a long time after Viete's 1591 introduction of the use of literal symbols as variables in algebra mathematicians used literal symbols to stand only for natural numbers. As a result, algebra was originally an algebra in which only natural numbers could be substituted for variables, or, in Peackock's words, an arithmetical algebra (Kline, 1980, p. 159.

According to Kline, it was Peacock's argument on "the principle of the permanence of equivalent forms" in 1833 that opened the way for mathematicians to fully accept the symbolic algebra of real numbers.

To sum up, we hypothesize that students start with an initial belief that literal symbols stand for natural numbers only and continue with some possible intermediate steps characterized by fragmentation or synthetic conceptions before they eventually understand that literal symbols can stand for any real number. Initial support for this hypothesis was obtained in a pilot study (Christou & Vosniadou, 2007) in which 13 students were interviewed on their understanding of the use of literal symbols in equations and inequalities. The students attended tenth grade (approximately 15 years of age) and had extensive instruction on the use of variables in equations and inequalities since eighth grade. They were asked questions such as whether there are some numbers that could be substituted for "b" so that "4b" would equal "3" (they were also presented with the formal equation "4b = 3"); whether a/b is always smaller than "1" (a/b < 1) in cases where "a" is smaller than "b" (a < b); and whether it is always true that "4b > 4/b"($< i > b < /i > \neq 0$). Many students interpreted the literal symbols as referring only to natural numbers, a tendency that resulted in specific mistakes. For example, in the first equation, 5 of the 13 students responded by substituting the literal symbols only with natural numbers, thus concluding that "4b" is always bigger than "4." Only three students gave the correct response, namely that there is certainly such a number, possibly a fraction or a decimal. The remaining five students solved the equation using the formal way.

Quite similar were the results regarding the inequalities. In the case of "a/b < 1" (given that a < b), only two of the students tried to solve the inequality, and the rest decided to check by substituting numbers for the literal symbols. Six of the 13 students tried only with natural numbers and another 4 used a negative integer as well. Only one student tried with at least one rational number. Similar results were obtained in the remaining cases. When the students were asked about the numbers that could be assigned to "4b" or "a/b", half the students responded by substituting only natural numbers for the literal symbols and their responses were multiples of "4" in the case of "4b" and positive fractions in the case of "a/b."

The three experiments presented in this article investigate in a more systematic way the hypothesis that the natural number bias influences students' substitutions of literal symbols in algebra.

EXPERIMENT 1

The first experiment tested the hypothesis that secondary school students who had been introduced to the concept of variable through systematic instruction would nevertheless exhibit a strong tendency to substitute literal symbols in algebraic expressions with natural numbers only. Because we had already conducted an initial individual interview pilot study with a small number of students (Christou & Vosniadou, 2007), we were interested in testing this hypothesis using a larger sample. For this reason, a group questionnaire methodology was used.

Considering the absence of previous relevant studies, the decisions regarding the questions used and the coding of the responses were guided by our theoretical considerations. The materials were validated through discussions with experts in mathematics education and researchers in the field of learning. They were also pilot tested with small groups of students, such as the pilot study mentioned previously.

Two open-ended questionnaires were constructed each consisting of the same seven algebraic expressions. In Questionnaire A (QR/A) the students were asked to indicate various possible numbers that could be assigned to the algebraic expressions. We hypothesized that a mathematically sophisticated person would understand that the expected response in QR/A was that "all numbers can be assigned to the literal symbols" in all the algebraic expressions.[2] Nevertheless, technically, any number—natural or non-natural—that the students would choose to substitute could be considered to be correct. For that reason a second questionnaire was designed, Questionnaire B (QR/B), in which the students were asked to indicate all numbers that could not be assigned to the algebraic expressions. In QR/B only one mathematically correct response applies, namely, that "there is no such number."

It was hypothesized that regardless of the type of questionnaire used, there would be a strong tendency on the part of the students to interpret literal symbols to stand for natural numbers only. It was also hypothesized that there will be intermediate steps in the process of understanding the concept of real variable, with the students producing internally inconsistent or synthetic responses.

METHOD

Participants

The participants were 57 students from two middle class, public high schools in Athens, Greece. Thirty-six (36) of these students attended eighth grade and 21 attended ninth grade (mean age 13.5 years). Twenty-nine (29) students completed QR/A, and 28 completed QR/B. The sample was almost equally divided between boys and girls (26 boys and 31 girls).

Materials

Both questionnaires presented the students with the following seven algebraic expressions: a, $-b$, $4g$, $1/g$, a/b, $d+d+d$, and $k+3$ (Q1 to Q7 respectively). The students were given the following instructions: "In algebra, we use literal symbols (such as a, b, x, y, etc.) to stand for numbers. In this questionnaire we use literal symbols in this way. Read the following questions carefully and respond with as many numbers as you can." In QR/A the students were asked to "Write down as many numerical values as you think can be assigned to . . ."; while in QR/B they were asked to "Write down as many numerical values as you think cannot be assigned to . . ."

Procedure

The two questionnaires were distributed randomly to the students in their classroom and were completed in the presence of their mathematics teacher and one of the experimenters. Only clarification questions were answered.

[2]With a few exceptions such as that a denominator cannot equal zero.

RESULTS AND DISCUSSION

Correct vs. Incorrect Responses

Both questionnaires showed high reliability (Cronbach's Alpha > 0.844). Students' responses in QR/A were originally categorized in three main categories: "Correct," "Incorrect," and "No Response." "Correct" were considered the responses "all numbers can be assigned to the algebraic expression" and those including both natural and non-natural numbers.[3] "Incorrect" were considered the responses that included only natural numbers or numbers of a specific kind, such as positive fractions. As can be seen in Table 1, the majority of the responses fell in the incorrect category. Only 23.6% of the total responses were placed in the correct category.

The same procedure was followed for QR/B. As shown in Table 1, 32.7% of students' total responses fell in the "Correct" category and 46.9% in the "Incorrect" category. The differences in students' performance between QR/A and QR/B were investigated using an independent sample t-test but were not found to be statistically significant, [$t(54) = 0.041, p = .323$]. In order to deal with a small deviation from the normal distribution, nonparametric tests were applied with the same results. The effects of other possible covariates such as gender or grade were examined using a regression analysis but were not found to be significant predictors of students' performance in the two questionnaires. Logarithmic transformation of the dependent variable was also applied and the analysis was run again showing the same results.

The finding that about 70% of the students gave erroneous responses, despite the fact that they had been exposed to extensive relevant instruction, clearly supports the hypothesis that there is a strong tendency to substitute natural numbers for the literal symbols.

Responses for the Algebraic Expressions "a" and "–b"

We continued with an examination of students' substitutions for the algebraic expressions "a" and "–b" in both questionnaires. We did this because they were the simplest expressions used in the questionnaires, containing only one variable. Although these two expressions had the same form, they differed in that a negative sign used in the second expression but not in the first.

Four types of mathematically incorrect responses were distinguished: (1) "Positive Integers" (natural numbers) such as 1, 2, 3, etc.; (2) "Negative Integers" such as -1, -2, -3, etc.; (3) "Positive

TABLE 1
Frequency and Percent of Students' Responses in Questionnaires A and B

Categories of Responses	*Questionnaire A*	*Questionnaire B*
Incorrect	138 (68.0%)	92 (46.9%)
Correct	48 (23.6%)	64 (32.7%)
No Response	17 (8.4%)	40 (20.4%)

[3]The response that zero cannot be substituted for "g" or "b" in the case of "1/g" and "a/b" respectively was also considered as correct.

Non-Integers" such as 2.3, 1/3, etc.; and (4) "Negative Non-Integers" such as -2.3, -1/3, etc. Positive and negative non-integer responses could include natural numbers and negative integers but should also include at least one positive or negative non-integer number respectively. The great majority of the erroneous responses in Questionnaire A fell in the positive integer category for the expression "a", and in the negative integer category for the expression "–b" (Table 2).

Naturally, the question arises as to whether the negative integer responses for –b represent truly negative numbers. If those responses were to be considered as representing negative numbers, then our hypothesis, namely that is there is a strong tendency in students to substitute literal symbols with natural numbers, would not be supported. However, we doubt that the same students, who used predominately natural numbers in all the other algebraic expressions, suddenly understood that in the case of "–b" they should use non-natural numbers. Rather, we believe that a different interpretation is more likely. More specifically, it can be argued that even in the case of "–b" the students substituted the literal symbol with natural numbers but retained the phenomenal sign of the variable—the negative sign. If we accept this interpretation, we come to the conclusion that not only there is a "natural number bias" as hypothesized but that there might also be a "phenomenal sign bias." The "phenomenal sign bias" is the tendency to consider the phenomenal sign of an algebraic expression—the external sign that a variable (or a simple algebraic expression which contains one or more variables) has as a superficial characteristic of its form—to be its real sign.

Given this, we categorized students' responses as (a) "Natural Number" (NN) or "Non-Natural Number" (nNN) depending on whether students substituted natural or non-natural numbers for the literal symbols, and (b) "Same Phenomenal Sign" (sPS) or "Different Phenomenal Sign" (dPS) depending on whether students' responses retained the phenomenal sign of the given expression or not. If this type of categorization is used, the differences between "a" and "–b" do not hold anymore. As can be seen in Table 2, both in the case of "a" and "–b" the great majority of the responses fell in the categories natural number and same phenomenal sign (NN/sPS). In other words, the students observed both the natural number bias and the phenomenal sign bias, thus responding with positive integers for "a" and negative integers for "–b" in QR/A. Interestingly, in QR/B, the majority of the responses fell in the category Natural

TABLE 2
Frequency and Percent of Students' Responses for the Algebraic Expressions "a" and "–b" in Questionnaires A and B

Categories of Responses	*Questionnaire A*		*Questionnaire B*	
	a	$-b$	a	$-b$
Positive Integers	19 (65.5%) [NN/sPS]	2 (6.9%) [NN/dPS]	1 (3.6%) [NN/sPS]	14 (50.0%) [NN/dPS]
Negative Integers	1 (3.5%) [NN/dPS]	21 (72.4%) [NN/sPS]	13 (46.4%) [NN/dPS]	1 (3.6%) [NN/sPS]
Positive Non-Integers	–	2 (6.9%) [nNN/dPS]	2 (7.1%) [nNN/sPS]	4 (14.3%) [nNN/dPS]
Negative Non-Integers	–	–	3 (10.7%) [nNN/dPS]	2 (7.1%) [nNN/sPS]
Correct	9 (31.0%)	4 (13.8%)	8 (28.6%)	5 (17.9%)
No Response	–	–	1 (3.6%)	2 (7.1%)

Note. NN: Natural Number; nNN: non-Natural Number; sPS: same Phenomenal Sign; dPS: different Phenomenal Sign.

Number/Different Phenomenal Sign (NN/dPS), which means that they changed the phenomenal sign of the algebraic expressions in QR/B but kept the natural number substitutions intact.

In order to further examine the differences between QR/A and QR/B statistically, we performed a set of chi square analyses. We compared students' combined responses in the two questionnaires based on whether they substituted the literal symbols with natural numbers or with non-natural numbers such as fractions or decimal numbers (Table 3). The analysis showed significant differences between the two questionnaires, [χ^2 *(3) = 11,922, p < .05*]. This difference could have resulted from the way the questions were phrased. However, as shown in Table 3, when the responses were categorized on a NN vs. nNN basis, in both questionnaires the dominant category of responses was the NN category, indicating that the majority of the students still tended to respond with numbers within the set of integers.

In a second analysis, we compared students' combined responses in the two questionnaires based on whether they used the same or different sign with the sign of the given algebraic expressions (see Table 3). There were significant differences in the two questionnaires [χ^2 (3) =49,675, p < .001]. Chi square analysis in the "Same Phenomenal Sign" (sPS) vs. "Different Phenomenal Sign" (dPS) categories of Table 3 indicated that the differences were due to students' tendency to observe the phenomenal sign bias in QR/A (category sPS) but not in QR/B (category dPS) [$\chi^2(1)=46{,}56$, *p < .001*].

The previously mentioned differences in the distribution of students' responses support our argument that there is a natural number and also a phenomenal sign bias in the ways students substitute numbers for the literal symbols. The same effect also appeared in the rest of the algebraic expressions used.

Analysis of Responses in the Remaining Algebraic Expressions

Students' responses in the remaining algebraic expressions were placed in the following six categories: "Correct," "Natural Number/Same Phenomenal Sign" (NN/sPS), "Natural Number/Different Phenomenal Sign" (NN/dPS), "Non-Natural Number/Same Phenomenal Sign" (nNN/sPS), "Non-Natural Number/Different Phenomenal Sign" (nNN/dPS), and "No Response." Examples of possible responses for each category are given in Table 4. The responses

TABLE 3
Frequency and Percent of Students' Responses for the Combined Algebraic Expressions "a" and "–b" When Categorized on Natural Number (NN) vs. Non-Natural Number (nNN) Basis or on Same Phenomenal Sign (sPS) vs. Different Phenomenal Sign (dPS) Basis in Questionnaires A and B

	Categories of Responses	*Questionnaire A*	*Questionnaire B*
Responses Categorized on NN vs. nNN Basis	NN	43 (74.1%)	29 (51.8%)
	nNN	2 (3.4%)	11 (19.6%)
Responses Categorized on sPS vs. dPS Basis	sPS	40 (69.0%)	6 (10.7%)
	dPS	5 (8.6%)	34 (60.7%)
	Correct	13 (22.4%)	13 (23.2%)
	No Response	0 (0.0%)	3 (5.4%)

Note. NN: Natural Number; nNN: non-Natural Number; sPS: same Phenomenal Sign; dPS: different Phenomenal Sign.

TABLE 4
Examples of Students' Responses in Each Category for Q3 to Q7 in Both Questionnaires A and B

Categories of Responses	*Algebraic Expressions*				
	4g	$\frac{1}{g}$	$\frac{a}{b}$	*d+d+d*	*κ+3*
NN/sPS	4·1, 4·2 1, 2, 3	$\frac{1}{2}, \frac{1}{3}$ 1, 2, 3	$\frac{2}{3}, \frac{3}{4}$	1+1+1 2+2+2 1, 2, 3	2+3, 3+3, numbers larger than 3
NN/dPS	4·(–1), 4·(–2)	$\frac{1}{-2}, \frac{1}{-3}$	$\frac{-2}{-3}, \frac{-3}{-4}$	(–1)+(–1)+(–1) (–2)+(–2)+(–2)	(–2)+3, (–3)+3
nNN/sPS	[2.3, $\frac{2}{3}$]	[2.3, $\frac{2}{3}$]	2.3, $\frac{2}{3}$	[2.3, $\frac{2}{3}$]	2.3, 3.7
nNN/dPS	–2.3, $\frac{2}{3}$	–2.3, $\frac{-2}{3}$	–2.3, $\frac{-2}{3}$	–2.3, $\frac{-2}{3}$	–2.3, $\frac{-2}{3}$

Note. NN: Natural Number; nNN: non-Natural Number; sPS: same Phenomenal Sign; dPS: different Phenomenal Sign.

placed in the square brackets represent possible expected responses, which, however, were not obtained in the actual sample.

One experimenter scored all the responses, while a second experimenter scored half of the responses using the same criteria. Agreement between the two scorers was 97%. All disagreements were discussed until consensus was achieved. Table 5 presents the frequencies and percent of students' responses in each of the above mentioned categories for the combined algebraic expressions Q3 to Q7, in QR/A and QR/B. In QR/A, 59.3% of the students substituted only natural numbers for the literal symbols and they also kept the phenomenal sign of the variable intact. In QR/B, the most frequent erroneous response was the NN/dPS response, meaning that the students continued to observe the natural number bias but not the phenomenal sign bias.

In order to examine statistically the differences between the two questionnaires we again compared the NN substitutions and the PS substitutions in the combined responses (see Table 6). Chi square analysis showed that there were statistically significant differences in students' responses in QR/A and QR/B [χ^2 *(3) = 37,681, p < .001*]. In QR/B there was an increase in nNN in

TABLE 5
Frequency and Percent of Students' Responses in Each Category for the Combined Algebraic Expressions (Q3–Q7) in Questionnaires A and B

Categories of Responses	*Questionnaire A*	*Questionnaire B*
NN/sPS	86 (59.3%)	4 (2.9%)
NN/dPS	–	30 (21.4%)
nNN/sPS	2 (1.4%)	3 (2.1%)
nNN/dPS	5 (3.4%)	15 (10.7%)
Correct	35 (24.1%)	51 (36.4%)
No Response	17 (11.7%)	37 (26.4%)

Note. NN: Natural Number; nNN: non-Natural Number; sPS: same Phenomenal Sign; dPS: different Phenomenal Sign.

TABLE 6
Frequency and Percent of Students' Responses for the Combined Algebraic Expressions (Q3–Q7) When Categorized on Natural Number(NN) vs. Non-Natural Number (nNN) Basis or on Same Phenomenal Sign (sPS) vs. Different Phenomenal Sign (dPS) Basis in Questionnaires A and B

	Categories of Responses	*Questionnaire A*	*Questionnaire B*
Responses Categorized on NN vs. nNN basis	NN	86 (59.3%)	34 (24.3%)
	nNN	7 (4.8%)	18 (12.9%)
Responses Categorized on sPS vs. dPS basis	sPS	88 (60.7%)	7 (5.0%)
	dPS	5 (3.4%)	45 (32.1%)
	Correct	35 (24.1%)	51 (36.4%)
	No Response	17 (11.7%)	37 (26.4%)

Note. NN: Natural Number; nNN: non-Natural Number; sPS: same Phenomenal Sign; dPS: different Phenomenal Sign.

"Correct" and also in "No Response" categories, apparently due to the differences in the way the questions were posed. Still, as shown in Table 6, students tended to respond with natural numbers or specific kinds of non-natural numbers such as integers but not with fractions or decimal numbers.

There were significant differences between students' responses in the two questionnaires when PS substitutions were compared [χ^2 *(3) =111,394, p < .001*]. A chi square analysis of the categories sPS vs. dPS in Table 6 indicated that the previously mentioned differences were due to students' tendency to retain the phenomenal sign in QR/A (sPS category) but to change it in QR/B (dPS category) [χ^2*(1)=97.25, p < .001]*. Students' tendency to change the phenomenal sign of the algebraic expressions in order to indicate the numbers that cannot be assigned for it in QR/B can be considered as an intermediate step in the process of understanding the concept of variable and further supports the interpretation that students consider the phenomenal sign of the algebraic expression to be its real sign.

EXPERIMENT 2

While the results of Experiment 1 supported our hypothesis, it could be argued that the students provided mostly natural numbers for the literal symbols because these are the most commonly used numbers, and therefore, they thought that this is what they were expected to do. This criticism cannot, of course, explain why some of the students chose to give the mathematically correct answer. It cannot also explain the systematic errors that the students made, such as the substitution of "–b" only with negative integer numbers or the exclusion of negative numbers as numbers that could be substituted for the positive-like algebraic expressions in QR/B.

Nevertheless, in order to further test our hypothesis, a second experiment was conducted using a multiple-choice instead of an open-ended questionnaire. In the multiple-choice questionnaire the students were presented with a set of responses that included an explicit version of the mathematically correct response. More specifically, the last alternative stated that "any number can be assigned to the algebraic expressions." In this task there could be no doubt as to what was the mathematically correct response. Questionnaire C (QR/C) used the same algebraic expressions as in the previous questionnaires. The negative form was used to pose the questions: "select the numbers that cannot be assigned to the algebraic expressions," because only under this condition (as in the case of QR/B in Experiment 1) there is an undisputed correct answer from a mathematical point of view.

It was hypothesized that students' interpretations of the literal symbols in algebra would be constrained both by the natural number bias and the phenomenal sign bias. It was also hypothesized that there would be intermediate steps of understanding, taking the form of fragmented or synthetic conceptions.

METHOD

Participants

The participants were 34 students from eighth and ninth grade attending the same two schools used in Experiment 1. The sample was almost equally divided between boys and girls (19 boys, 15 girls) and eighth and ninth graders (18 eighth and 16 ninth graders), and had a mean age of 13.5 years.

Materials

The students were presented with the same algebraic expressions as in the first study, namely the expressions: a, –b, 4g, a/b, d+d+d, and k+3 (Q1 to Q6), with the exception of expression "1/g," which was excluded because in Experiment 1 the students responded to it in a similar way as in "a/b." The students were given the following instructions: "In algebra, we use literal symbols (such as a, b, x, y, etc.) mostly to stand for numbers. In this questionnaire we use literal symbols this way. Read the following questions carefully. If you think there are some numbers among the given alternatives that you think cannot be assigned to the given algebraic expressions, please place a circle around them. You may choose more than one number if you wish."

The students were given a set of alternatives consisting of 11 number choices, which included positive and negative fractions, positive and negative decimals, and positive and negative integers. The twelfth alternative was always the correct response, namely, that "No, all numbers can be assigned to it."

Procedure

The same procedure was used as in the previous experiment.

RESULTS AND DISCUSSION

Despite the relatively small sample, there was a big effect size between correct (18.6%) and incorrect responses (81.4%), supporting our hypothesis that the students would tend to think of specific numbers only as possible substitutions for the literal symbols. This finding indicates that even students who may have understood that literal symbols in algebra stand for the generalized number may not know that literal symbols can stand for any real number.

In order to further examine our hypothesis we focused the analysis on the numbers from the given set that the students tended to exclude as numbers that could not be assigned to the given expressions. Students' responses in the first two questions ("a" and "–b") were grouped into the five categories presented in Table 7. The "Correct" category included only the correct response. The "Natural Number" category was divided in two subcategories: (1) "Natural Number Only" (NNO), which excluded all numbers except those that would have come up if only natural numbers were substituted for the literal symbols, and (2) "Specific non-Natural Number" (SNN), which excluded only a specific category of non-natural numbers (for example, all fractions in the case of "a", but not the decimals). The "Phenomenal Sign" category included responses that excluded numbers based on the positive/negative sign distinction. The "Non-Systematic" category included responses that lacked any obvious systematicity.

One of the experimenters scored all the responses while the second scored half of the responses using the same criteria. Agreement between the two scorers was 96%. All disagreements were discussed until consensus was achieved.

As can be seen in Table 7, only a few students chose the mathematically correct response even though it was explicitly stated in the questionnaire as an alternative. Most of the students excluded specific numbers from the given set. Chi square analysis showed no statistically significant differences between students' responses in expressions "a" and "–b" [χ^2 *(5)* $=$ *8.45,* $p =$ *.133*]. In both cases the responses were consistent with the belief that literal symbols stand for natural numbers or that the phenomenal sign of the expressions must be retained.

Responses in the remaining algebraic expressions were categorized in a similar way. Table 7 presents the frequency and percent for each category and Table 8 shows examples of students' responses for each category.

Again, only a few students gave the correct response. The erroneous responses fell mostly in the "Natural Number" category and fewer in the "Phenomenal Sign" category. There were, however, a relatively large number of non-systematic responses, possibly due to the complexity and counter-intuitiveness of the questions in QR/C and the fact that they were expressed in the negative form. Those responses could be explained as fragmented or synthetic responses

TABLE 7
Frequency and Percent of Students' Responses for the Algebraic Expressions in Questionnaire C

		Algebraic Expressions		
Categories of Responses		*a*	$-b$	*Rest Algebraic Expressions Combined*
Natural Numbers	NNO	7 (20.5%)	2 (5.9%)	36 (26.5%)
	SNN	3 (8.8%)	2 (5.9%)	12 (8.8%)
Phenomenal Sign		13 (38.2%)	18 (52.9%)	21 (15.4%)
Non-Systematic		1 (2.9%)	5 (14.7%)	27 (19.9%)
Correct		10 (29.4%)	6 (17.6%)	22 (16.2%)
No Response		0 (0.0%)	1 (2.9%)	18 (13.2%)

Note. NNO: Natural Numbers Only; SNN: Specific non-Natural Numbers.

TABLE 8
Examples of Students' Responses for Algebraic Expressions (Q3–Q6) in Questionnaire C

		Algebraic Expressions			
Categories of Responses	*4g*	$\frac{a}{b}$	*d+d+d*		κ*+3*
Natural Numbers	NNO	all but natural numbers	all but positive fractions	all but natural numbers	all but natural numbers
	SNN	fractions	integers	fractions	fractions
Phenomenal Sign		negatives	negatives	negatives	Negatives

Note. NNO: Natural Numbers Only; SNN: Specific Non-Natural Numbers.

representing intermediate steps of understanding between the belief that literal symbols can stand only for natural numbers and understanding that any real number can be a possible substitute. In addition, some students appeared willing to lift the natural number constraint but only in a limited fashion. These students accepted that literal symbols can stand, in addition to natural numbers, for specific types of non-natural numbers only, such as the decimal numbers (the SNN category).

EXPERIMENT 3

Experiment 3 investigated the same hypotheses using an individual interview methodology and by placing the algebraic expressions in a more familiar mathematical domain, namely that of inequalities. An individual interview makes it possible to better examine the strength of students' tendency to substitute natural numbers for the literal symbols and their willingness to change when provided with appropriate hints.

In this experiment the students were presented with algebraic expressions in algebraic inequalities. They were asked to validate six inequalities that expressed relations between algebraic expressions and to say whether these relations are true for any number that could be substituted for the literal symbols. In order to encourage the students to check the validity of the inequalities by substituting numbers and not by trying to solve the inequalities formally, the questions were phrased in ways that evoked the semantic meaning of the inequalities as relations between number quantities. We deliberately did not use the usual way such questions are posed in the school context (i.e., "find the solution of the inequality"). For the same reason, the inequalities used were specifically designed to make formal solutions more difficult compared to following the strategy of trying with numbers.

It was hypothesized that the students would again try to substitute the literal symbols only with natural numbers and that this tendency would result in specific mistakes in the given questions. However, we were also interested in finding out the strength of this tendency by giving hints to the students to try with different kinds of numbers and by encouraging them to think of non-natural number substitutions for the literal symbols. For this reason, the following two hints were used:

- Hint 1 (h1): Are you certain about your response? Would you like to try with another (number)?

- Hint 2 (h2): Do you think this is true (or not true) for every number that you know? Would you like to try with another kind of number? Could you find a number which could change your initial response?

It was hypothesized that these two hints would help students think of non-natural numbers in their substitutions but that they would not totally eliminate the effects of the natural number bias.

METHOD

Participants

The participants were eight students attending tenth grade (mean age 15 years) in the same schools used in the previous experiments. The sample was equally divided between boys and girls. The students had been taught the formal way of solving inequalities and had at least two years of experience using real variables in functions, equations, and inequalities.

Materials

An open-ended questionnaire (QR/D) was designed, which included six questions that concerned the validity of algebraic inequalities. The questions were posed as follows: "Is it always true that 5d > 4/d, regardless of the number that could be substituted for d?"

The inequalities used were the following: $5\text{d} > 4/\text{d}$ ($d \neq 0$)*, $7\text{a} < 6/\text{a}$ ($a \neq 0$), $\text{a}+1>1/\text{a}$ ($a \neq 0$) *, $9\text{x} < 3 + 1/\text{x}$ ($x \neq 0$), $-\text{k}-1>2/\text{k}$ ($k \neq 0$) , $-\text{y} < \text{y}+1/\text{y}$ ($y \neq 0$)*. They were specifically designed to be correct or incorrect for the small set of natural numbers that we expected students to substitute in order to check their validity. Half of the inequalities (those noted with an asterisk) were valid for the first series of natural numbers (i.e., 1, 2, . . .) while the rest were invalid for those numbers. The correct response was that "the validity of the inequality depends on the numbers used; there are some numbers for which the inequality is true and others for which it is false."

If the students checked the validity of the inequalities with an asterisk only with natural numbers, they would erroneously conclude that they were valid and that the remaining inequalities were invalid. When this happened we continued the interview giving the hints mentioned earlier.

Procedure

The students were given the questions one by one, in the same order. They were given some time to read the question, respond with yes or no, and explain why. If they gave an erroneous response the interviewer continued with the hints in the order presented earlier.

TABLE 9
Students' Final Responses in Each of the Given Inequalities in Experiment 3

	Inequalities Used					
Participants	$5d>4/d$	$7a<6/a$	$a+1>1/a$	$9x<3+1/x$	$-k-1>2/k$	$-f<f+1/f$
P1	IN	IN	IN	IN	IN	IN
P2	NN	NN	NN	NN	NN	NN
P3	NN	NN	NN	NN	IN	IN
P4	RN	RN	RN	RN	RN	RN
P5	NN	NN	NN	NN	NN	IN
P6	NN	NN	NN	IN	IN	IN
P7	NN	NN	NN	NN	NN	IN
P8	NN	NN	NN	NN	NN	NN

Note. NN: Natural Numbers; IN: Integers; RN: Rational Numbers.

RESULTS AND DISCUSSION

The kinds of numbers students used for their final substitutions[4] (after the hints) were grouped in the categories: "Natural Numbers" (NN) if they used only natural numbers; "Integers" (IN) if they used at least one negative integer; and "Rational Numbers" (RN) if they used at least one rational number (Table 9).

Only one of the students (P4) responded to all the questions using at least one rational number, which was always a positive decimal number. This student concluded that the given inequalities are not always valid or invalid for any number but that their validity depends on the kind of numbers used. The responses of this student were the closest to the correct response.

Five of the eight students insisted on checking with natural numbers even after the first hint from the interviewer. However, after the second hint, they tried with at least one negative integer (Integer category) in some of the given questions. They never tried, however, with a rational number.

Two students (P2 and P8) responded to the first two questions without even checking with numbers. They argued that $5d > 4/d$ is always true because multiplying any numbers by 5 would always make bigger than dividing 4 by this number. They also argued that multiplying a number with 7 would never be smaller than 6 divided by this number, and thus they said that the inequality $7a < 6/a$ would never be true. Following the interviewers' hints to check their conclusion with some other numbers, they continued to check only with natural numbers, which vouched for their initial response. These students never tried with any non-natural number despite the first two hints provided by the interviewer.

The following exchange is representative of the way one of the students reacted to the suggestions by the experimenters to think of other kinds of numbers in his substitution.

[4]Only one of the students tried to solve the first two inequalities formally but she did not succeed and accepted the interviewers' suggestion to try with numbers.

John (tenth grader, 14.5 years old)
Exp: In the question "Is it always true that 5 < *i* > *d* < /*i* >> 4/ < *i* > *d* < /*i* > for any number d?" you responded that it is always true. How did you think of that?
John: If we substitute 2 for b, 2*5 = 10 and 4:2 = 2, so it (the inequality) is valid.
Exp: Do you think it is valid for any number?
John: Yes.
Exp: Would you like to try with another number?
John: Ok, let's try with 3, 15 > 4/3, so it is valid.
Exp: Would it be valid for all the numbers that you know?
John: Mmm! Yes.
Exp: Would you like to try with another kind of number?
John: Let's try with 4, 20 > 1, also valid.
Exp: Do you think it would be possible to find a number which wouldn't hold for the inequality?
John: I don't believe we could. No.

At the end of the interview, the interviewer explicitly suggested to these two students who insisted on trying only with natural numbers throughout the interview, to try with non-natural numbers: "Could we try with a negative number, let's say –2?" and "Could we also try with a fraction, let's say $\frac{1}{2}$?" Both students agreed that it would be possible. An excerpt from the continuation of the interview with the student we call John is given next.

Exp: Could we try with a negative number; let's say –2?
John: Well, yes, of course. Mmm! that is –10 > –2, which is not valid, well, this changes everything.
Exp: So, now do you think we could find a number which wouldn't hold for the inequality?
John: Certainly, the negative numbers.
Exp: Any other kind of number?
John: We have tried everything, positives and negatives.
Exp: Could we try with a fraction; let's say $\frac{1}{2}$?
John: Yes, we could [he tries it]. This also wouldn't hold.
Exp: Why you didn't try with such kind of numbers before?
John: I don't know, I should have thought both of negatives or fractions.

The finding that the two students who responded only with natural number substitutions throughout the interview were willing to accept non-natural numbers as possible substitutes when the interviewer explicitly suggested this shows that they were familiar with the idea that literal symbols in algebra could stand for non-natural numbers. Nevertheless, there was a natural number bias which continued to influence their substitutions, even after the hints, in the absence of direct external input.

A quantitative analysis of the total 48 responses (8 students, 6 responses each) indicated that the majority of the substitutions (29/48 or 60%) involved only natural numbers. Twenty-eight percent of the responses (13/48) involved at least one negative integer number. As it appears in Table 9, in all those cases the students initially tried with natural numbers only, but after the second hint, they tried with a negative integer in some of the questions. They also tried substitutions with at least one negative integer in the last two inequalities, which contained algebraic expressions with a negative phenomenal sign (i.e., $-k-1 > 2k$). These findings further support the hypothesis that when students substitute numbers for the literal symbols they tend to think of numbers within the set of integers rather than with fractions or decimals. They also demonstrate the influence of the phenomenal sign bias.

GENERAL DISCUSSION

The results of the three experiments can be summarized in the following main findings. First, students tended to substitute mostly natural numbers for the literal symbols, both when they appeared in algebraic expressions and in a more familiar mathematical domain, such as that of inequalities. This was true for open-ended and multiple-choice tasks and appeared also in individual interviews where the experimenter encouraged the students to think with other kind of numbers. We called this tendency the "natural number bias." This finding has not been reported in the literature before, and we believe it adds to the existing literature. It shows that even when students realize that literal symbols in algebra can stand for the generalized number they may mean any natural number rather than any real number.

Second, there was a strong tendency to substitute the algebraic expression "–b" with negative integers, while algebraic expressions that did not contain a negative sign were considered to represent only positive values. This finding indicates that students interpreted the phenomenal sign of the variables to be the actual sign of the algebraic expressions. We called this tendency the "phenomenal sign bias." Some other researchers have also noted a similar phenomenon (Chiarugi, Fracassina, & Furinghetti, 1990; Crowley et al., 1994; Gallardo, 2002; Vlassis, 2004).

It is possible to explain this finding as a byproduct of the natural number bias. That "no sign" implies positive value and minus sign means "negative value" is a characteristic of numbers in arithmetic. If students think that literal symbols should be substituted with natural numbers, then it is natural to conclude that the phenomenal sign of a variable is its "real" sign. The phenomenal sign bias could also be obtained from the interpretation of literal symbols to stand for positive rational numbers. This is a less likely interpretation, however, given that substitutions with rational numbers were very rare.

Third, in addition to substituting "–b" with negative integers there was a strong tendency in students to respond with integer numbers with a different sign from the phenomenal sign of the given expressions when asked which numbers cannot be assigned for "–b" in QR/B. These kinds of responses allowed students to retain their natural numbers substitutions but also change something, namely the phenomenal sign, in response to the negative demands of the questions. Indeed this type of response can be interpreted as an intermediate step—as a synthetic conception—in the process of understanding that literal symbols can be substituted with any real number.

Students showed some other intermediate steps in this process, where they accepted some but not all kinds of non-natural numbers as possible substitutes, especially in Questionnaire C. The kinds of non-natural number substitutes accepted appeared to depend on the kind of questions asked and the form of the algebraic expression used. Some students in Experiment 2 accepted decimals but not fractions for some of the cases, or fractions but not decimals for other. Some students were willing to accept non-natural numbers substitutes as long as they did not violate the phenomenal sign of the algebraic expressions. It was not possible due to the small sample size to determine how systematic these responses were—they could be characterized either as synthetic or as fragmented. Regardless of their exact nature, they reveal the presence of intermediate steps in the process of moving beyond only natural number substitutions without having fully acquired the mathematical concept of the real variable. It is of course important to investigate this phenomenon further using larger populations and in a greater variety of mathematical tasks.

These results support the predictions of the framework theory that there will be a strong tendency in students to substitute literal symbols with natural numbers, and that the process of developing the concept of real variable will be characterized by intermediate steps of understanding. Nevertheless, there are a number of alternative interpretations of the findings that also need to be examined.

It could be argued, for example, that the students provided mostly natural numbers for the literal symbols by default, since these are the most commonly used numbers and, therefore, that the natural number and the phenomenal sign bias are not robust and can be easily overcome. Such an interpretation could possibly explain why students substituted mostly natural numbers in Questionnaire A but could not explain why they excluded non-natural numbers in Questionnaire B or in the multiple-choice Questionnaire C, where the correct response was one of the alternatives. Over and above it could not explain students' tendency to try only with natural numbers in the interview study, despite the interviewers' consecutive hints to try with different kinds of numbers. Students' repeated reluctance to respond with non-natural numbers in the three experiments, which used different methodologies, indicates that this is a prevalent phenomenon. The fact that some students accepted the interviewers' explicit suggestion to try with a rational number in Experiment 3 does not, we believe, reduce the severity of the natural number and the phenomenal sign bias. It only affirms that the students had been exposed to non-natural number substitutions and that they were willing, in principle, to check with non-natural numbers when told to do so.

Additional support for our hypothesis comes from an intervention study in which we attempted to make students aware of the natural number and the phenomenal sign bias (Christou & Vosniadou, in preparation). The students were explicitly told that they should always try with at least one negative number when they needed to validate the sign of an expression and were provided with counterexamples creating cognitive conflict. Immediately after the intervention the students appeared less likely to show the phenomenal sign bias, but only for the set of tasks that were used as examples in the intervention. The majority of them did not transfer this knowledge to tasks that concerned inequalities and absolute values that were not used in the intervention as examples. Furthermore, the profits were not maintained one month later, suggesting that both the natural number bias and the phenomenal sign bias are deep rooted and difficult to overcome.

It could be argued that students' tendency to substitute only natural numbers for literal symbols could be an artifact of Greek students' experiences in school. It is true that mathematics curriculum materials use natural numbers exceedingly, in examples, implementation of rules, coefficients, or even in solutions of equations, etc., strengthening the natural number bias (see also Greer, 2006), and this applies to the Greek curricula also. Nevertheless, the Greek students, who are introduced to algebra in the seventh grade and have acquired extensive experience using literal symbols in a variety of algebraic tasks by tenth grade, are explicitly instructed to use as many different kinds of numbers as possible when substituting numbers for variables in order to draw the graph of a function, when looking for counterexamples in determining the validity of square root functions, when calculating the absolute values of algebraic expressions, or when making transformations for solving algebraic inequalities.

Further support for the argument that the natural number bias is not an artifact of the Greek curriculum comes from a replication study in which our multiple-choice questionnaire, Questionnaire C, was administered to 128 Flemish ninth graders. It should be mentioned here that Flemish students were placed first in mathematics in the 2003 Programme for International Student Assessment (PISA, 2003). Although the Flemish students were less susceptible to the natural number bias than the Greek students, 76% of them excluded some of the numbers in the multiple-choice questionnaire from being assigned to literal symbols at least in one of the algebraic expressions (Christou, Vamvakoussi, & VanDooren, 2010).

Finally, it could be argued that students do not substitute literal symbols with fractions because they think of fractions as pairs of whole numbers and not as single numbers (Mack, 1993, 1995; Streefland, 1991). We agree with such an interpretation, which we think is in line with the framework theory approach. According to the framework theory, students do not think of fractions as numbers with the same status as natural numbers because their beliefs about what counts as a number stand in the way of understanding natural and non-natural numbers within a unified system of real numbers (see also Vamvakoussi & Vosniadou, 2010).

To sum up, the results of the three experiments reported in this article reveal that students face a number of interrelated difficulties in substituting numbers for literal symbols. We have argued that these difficulties stem from a systematic influence of the natural number bias rather than from occasional intrusions of prior knowledge, or from other incidental factors, supporting the framework theory approach. These results suggest that many students are initially restricted to an algebra of natural numbers, similar in some respects to the historical case (see Kline, 1980).

This restriction has important implications for students' performance in many other mathematical tasks. For example, students often interpret graphs as discrete points and ascribe no meaning to line segments connecting those points (Leinhardt et al., 1990). This is probably due to their tendency to interpret variables as symbols that represent a series of natural numbers, which correspond to discrete points in the x-axes. As mentioned earlier, preliminary research in our lab supports this interpretation (Christou & Vosniadou, in preparation). Students who misinterpret literal symbols to stand for natural numbers only can also have difficulties with algebraic transformations in equations and inequalities and with calculating the absolute value of algebraic expressions (Chiarugi et al., 1990).

For students to welcome themselves to the "real" world of algebra, they need to develop the number concept beyond the constraints of the initial framework of numbers, which is grounded on natural numbers (see also Greer, 2006). In other words, developing the mathematical concept of real variable may require understanding the concept of rational number in arithmetic, something that few students have achieved by the time they are exposed to variables. It is also possible, however, that the development of a mathematically sophisticated conception of the use of literal symbols in algebra could support a deeper understanding of the number concept (Schmittau, 2005). It would certainly be interesting in future research to compare students' understanding of real numbers before and after they are introduced to the concept of variable in order to better investigate the possible interactions between these two factors.

The findings reported in this article are generally consistent with and further enrich existing mathematics learning theories. For example, Schoenfeld and colleagues (1993) provided a very detailed analysis of a single students' acquisition of graphs and equations of simple algebraic functions in the Cartesian plane based on the description of an architecture of four levels of knowledge structures, with "knowledge schemata" being the highest level of organization (macro-organization) and contextual primitive elements (such as diSessa's p-prims; see 1993, 2008) at the lowest level of organization. They then proceeded to show in a very detailed analysis the evolution of the participant's knowledge structures as revealed by her performance over a period of seven hours in seven weeks. One of the important conclusions of their paper is that the higher level structures constrain in very important ways the knowledge acquisition process. Here is what they said:

> We have outlined the benefits of a tightly connected (and correct) knowledge structure. It is, we noted, resistant to change: Knowledge that does not fit the structure has a hard time taking hold, and knowledge that does fit is resilient and tends to be regenerated. In fact, this is true whether or not the knowledge elements and connections in a tightly connected structure are correct. In IN's case, although many of the knowledge elements were unstable and many of the connections among them were tenuous, there were enough connections to import some degree of stability and resilience to her knowledge. The implications of her having this structure were extremely important. It shaped what IN perceived in the domain, what she attended to, and what she learned (i.e., which knowledge elements were created, modified, strengthened, or weakened, and which connections she made). (p. 137)

We completely agree with this, with one important exception, namely, that the construct "schema" represents the highest level knowledge structure in this hypothetical conceptual analysis. We argue that it is more accurate to assume that schemata are embedded within even larger belief networks, which we call "framework theories." A framework theory is not, of course, an explicit, well-defined, and socially shared scientific theory. The natural number concept in arithmetic appears to meet the criteria to count as such a framework theory. In our view, the erroneous schema structure that constrained IN's knowledge acquisition in the Schoenfeld and associates' (1993) study was possibly a synthetic conception, which caused further fragmentation and erroneous conceptions down the road.

We would like to clarify once more that our position with respect to the fragmentation perspective (diSessa, 1988, 1993) is not that fragmentation does not exist and that students move from one coherent "theory" to another. On the contrary, we argue that there is a great deal of fragmentation and instability, which is produced because of the incompatibility between the new information and students' prior knowledge, the complexity of the required changes, and the additive nature of the learning mechanisms employed.

Finally, as discussed in the introduction, we agree with Sfard's (1991) analysis of the formation of a new structural conception in mathematics. We think that it is useful to analyze students' difficulties with the number concept in the transition from arithmetic to algebra as requiring an ontological shift. Sfard (1991) does not discuss the problem of misconceptions in the transition from one structural conception to another, but it is natural to assume that when students are taught new procedures that depend on a different structural conception from the one they have, they may create various systematic errors in the process.

In general, the framework theory approach to conceptual change focuses on learning in cases where well-established prior knowledge can stand in the way of learning new concepts and where learning may require the radical revision of existing ideas. We believe that foregrounding such cases in mathematics learning is an important point to make because mathematics teaching has often underestimated the amount of knowledge revision that may be required in the process of acquiring mathematical expertise. Although we believe that it is important to alert students to the fact that some aspects of their prior knowledge may not be always consistent with the new, to-be-acquired information, we also think that it is necessary to try to locate elements of prior knowledge that can be used constructively to build new learning (see also Resnick, 2006; Vosniadou et al., 2008). A number of projects, such as the Early Algebra project (see for example Carraher et al., 2001; Carraher, Schliemann, & Schwartz, 2007), examine possible ways to build on students' prior knowledge of numbers and arithmetic to assist then to develop algebraic reasoning (see also Carpenter & Fennema, 1992; Carpenter, Fennema, & Franke, 1996; Kaput, 2001; Kaput & Blanton, 2005).

Other researchers have suggested ways to foster students' generalization abilities (see for example, Fujii, 2003), or use computer based environments, spreadsheets, and graphic calculators to help students create meaning for algebraic notation and understand algebraic operations (Cedillo & Kieran, 2003; Dettori, Garuti, & Lemut, 2001; Filloy, Rojano, & Rubio, 2003; Graham & Thomas, 2000). From our perspective, these suggestions could easily be enriched to include non-natural numbers in the various activities in order to help students consider different types of numbers as possible substitutes for literal symbols. Students could also be helped to expand their conceptual fields beyond natural numbers by using quadratic equations of the sort $2.67x^2 - 3.86x - 12.23 = 0$, which could be easily solved with the use of a calculator (Greer, 2006).

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REFERENCES

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For the Learning of Mathematics*, *14*(3), 24–35.

Asquith, P., Stephens, A. C., Knuth, E. J., & Alibali, M. W. (2007). Middle school mathematics teachers' knowledge of students' understanding of core algebraic concepts: Equal sign and variable. *Mathematical Thinking and Learning*, *9*(3), 249–272.

Behr, M. J., Harel, G., Post, T. R., & Lesh, R. (1992). Rational number, ratio, and proportion In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296–333). New York, NY: Macmillan.

Booth, L. R. (1984). *Algebra: Children's strategies and errors*. Windsor, Berkshire, UK: NFER-Nelson.

Booth, L. R. (1988). Children's difficulties in beginning algebra *The ideas of algebra, K-I2 (1988 Yearbook)* (pp. 20–32). Reston, VA: National Council of Teachers of Mathematics.

Carpenter, T. P., Corbitt, M. K., Kepner, H. S. J., Lindquist, M. M., & Reys, R. E. (1981). *Results from the second mathematics assessment of the national assessment of educational progress*. Reston, VA: National Council of Teachers of Mathematics.

Carpenter, T. P., & Fennema, E. (1992). Cognitively guided instruction: Building on the knowledge of students and teachers. *International Journal of Educational Research*, *17*, 457–470.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal*, *97*, 3–20.

Carraher, D. W., Schliemann, A., & Brizuela, B. (2001). *Can young students operate on unknowns?* Paper presented at the 25th Conference of the International Group for the Psychology of Mathematics Education, Utrecht, The Netherlands. Q3

Carraher, D. W., Schliemann, A. D., & Schwartz, J. (2007). Early algebra is not the same as algebra early. In J. Kaput, D. W. Carraher & M. Blanton (Eds.), *Algebra in the early grades* (pp. 235–272). Mahwah, NJ: Lawrence Erlbaum.

Cedillo, T., & Kieran, C. (2003). Initiating students into algebra with symbol-manipulating calculators. In J. T. Fey (Ed.), *Computer algebra systems in secondary school mathematics education* pp. 219–239. Reston, VA: National Council of Teachers of Mathematics.

Cerulli, M., & Mariotti, M. A. (2001). L'Algebrista: A Microworld for symbolic manipulation. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th IGMI Study Conference: The Future of Teaching and Learning of Algebra* (pp. 179–186). Melbourne, Australia: The University of Melbourne.

Chiarugi, I., Fracassina, G., & Furinghetti, F. (1990). *Learning difficulties behind the notion of absolute value*. Paper presented at the 13th Conference of the International Group for the Psychology of Mathematics Education, Mexico. Q4

Christou, K. P., Vamvakoussi, X., & VanDooren, W. (2010). *Students' ways to interpret literal symbols in algebra–A Comparison between Greek and Flemish students*. Paper presented at the 7th Biennial Meeting on Conceptual Change, Leuven, Belgium. Q5

Christou, K. P., & Vosniadou, S. (2007). *Students' interpretations of algebraic expressions in inequalities*. Paper presented at the 12th Biennial Conference for Research in Learning and Instruction, Budapest, Hungary. Q6

Christou, K. P., & Vosniadou, S. (in preparation). Literal symbols as natural numbers and the phenomenal sign of algebraic expressions. Q7

Collis, K. F. (1975). *The development of formal reasoning*. Newcastle, Australia: University of Newcastle.

Crowley, L., Thomas, M., & Tall, D. (1994). *Algebra symbols, and translation of meaning*. Paper presented at the 8th International Conference for the Psychology of Mathematics Education Lisbon, Portugal. Q8

Demby, A. (1997). Algebraic procedures used by 13- to 15-year-olds. *Educational Studies in Mathematics*, *33*, 45–70.

Dettori, G., Garuti, R., & Lemut, E. (2001). From arithmetic to algebraic thinking by using spreadsheet. In R. Sutherland, T. Rojano, A. Bell & R. Lins (Eds.), *Perspectives on school algebra* (pp. 191–207). Dordrecht, The Netherlands: Kluwer.

diSessa, A. A. (1988). Knowledge in pieces In G. Forman & P. Pufall (Eds.), *Constructivism in the computer age* (pp. 49–70). Hillsdale, NJ: Lawrence Erlbaum.

diSessa, A. A.. (1993). Toward an epistemology of physics. *Cognition and Instruction*, *10*, 105–225.

diSessa, A. A. (2008). A bird's-eye view of the "pieces" vs. "coherence" controversy (from the "pieces" side of the fence). In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (pp. 35–60). New York, NY: Routledge.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95–123). Dordrecht, The Netherlands: Kluwer Academic.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics*, *9*(2), 19–25.

Filloy, E., Rojano, T., & Rubio, G. (2003). Propositions concerning the resolutions of arithmetical-algebraic problems. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on School Algebra* (pp. 155–176). Dordrecht, The Netherlands: Kluwer.

Fischbein, E., Deri, M., Nello, M., & Marino, M. (1985). The role of implicit models in solving problems in multiplication and division. *Journal of Research in Mathematics Education*, *16*, 3–17.

Fujii, T. (2003). Probing students' understanding of variables through cognitive conflict problems: Is the concept of variable so difficult for students to understand? In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* Vol. 1, pp. 49–65. Honolulu, HI. Q9

Gallardo, A. (2002). The extension of the natural number domain to the integers in the transition from arithmetic to algebra. *Educational Studies in Mathematics* (49), 171–192.

Gelman, R. (2000). The epigenesis of mathematical thinking. *Journal of Applied Developmental Psychology*, *21*, 27–37.

Graham, A. T., & Thomas, M. O. J. (2000). Building a versatile understanding of algebraic variables with a graphic calculator. *Educational Studies in Mathematics*, *41*, 265–282.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: A perceptual view of simple arithmetic. *Journal of Research in Mathematics Education*, *25*(2), 115–141.

Greer, B. (2006). *Designing for conceptual change*. Paper presented at the 30th conference of the international group for the psychology of mathematics education, Prague, Czech Republic. Q10

Harel, G., Behr, M., Post, T., & Lesh, R. (1994). The impact of number type on the solution of multiplication and division problems: Further considerations. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 365–388). Albany, NY: SUNY Press.

Hart, K. M. (1981). *Children's understanding of mathematics*: 11–16. London, UK: John Murray.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, *27*, 59–78.

Johanning, D. I. (2004). Supporting the development of algebraic thinking in middle school: A closer look at students' informal strategies. *Journal of Mathematical Behavior*, *23*, 371–388.

Kaput, J. (1989). Linking representations in the symbol system of algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 167–194). Reston, VA: Lawrence Erlbaum, National Council of Teachers of Mathematics

Kaput, J. (2001). *Employing children's natural powers to build algebraic reasoning in the content of elementary mathematics*. Mahwah, NJ: Lawrence Erlbaum.

Kaput, J., & Blanton, M. (2005). The several forms of algebraic reasoning and their connections to elementary mathematics. In J. Kaput, D. W. Carraher, & M. Blanton (Eds.), *Algebra in the early grades*. Hillsdale, NJ & Reston, VA: Erlbaum & the National Council of Teachers of Mathematics. Q11

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York, NY: ______. Q12

Kieran, C. (2007). Learning and teaching of algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707–762). Reston, VA: National Council of Teachers of Mathematics.

Kline, M. (1980). *Mathematics, the loss of certainty*. New York, NY: Oxford University Press.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2005). Middle school understanding of core algebraic concepts: Equivalence and variable. *International Reviews on Mathematical Education*, *37*(1), 68–76.

Kuchemann, D. (1978). Children's understanding of numerical variables. *Mathematics in School*, *7*(4), 23–26.

Kuchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics: 11–16* (pp. 102–119). London, UK: John Murray.

Lee, L., & Wheeler, D. (1981). The arithmetic connection. *Educational Studies in Mathematics*, *20*, 41–54.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, *60*(1), 1–64.

Linchevski, L., & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: operating on the unknown in the context of equations. *Educational Studies in Mathematics*, *30*, 39–65.

MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation: 11–15. *Educational Studies in Mathematics*, *33*(1), 1–19.

Mack, N. K. (1988). Learning fractions with understanding: Building upon informal knowledge. *Journal for Research in Mathematics Education*, *21*, 16–32.

Mack, N. K. (1993). Learning rational numbers with understanding: The case of informal knowledge. In T. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 85–105). Hillsdale, NJ: Lawrence Erlbaum.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education*, *26*, 422–441.

Malara, N. A. (1999). *An aspect of long-term research on algebra: The solution of verbal problems*. Paper presented at the 23rd Conference of the International Group for Psychology of Mathematics Education, Haifa, Israel. Q14

Malara, N. A., & Iaderosa, R. (2000). The interweaving of arithmetic and algebra: Some questions about syntactic, relational and structural aspects and their teaching and learning In E. Swanke (Ed.), *European research in mathematics education* Vol. 2, (pp. 159–171). Q13

Matz, M. (1980). Building a metaphoric theory of mathematical thought. *Journal of Mathematical Behavior*, *3*(1), 93–166.

Moss, J. (2005). Pipes, tubes, and beakers: New approaches to teaching the rational-number system. In S. Donovan & J. D. Bransford (Eds.), *How students learn: History, mathematics, and science in the classroom* (pp. 309–349). Washington, DC: National Academy Press.

Nathan, M. J., & Koellner, M. (2007). Introduction: A framework for understanding and cultivating the transition from arithmetic to algebraic reasoning. *Mathematical Thinking and Learning*, *9*(3), 179–192.

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, *40*(1), 27–52.

Nickson, M. (2000). *Teaching and learning mathematics: A teacher's guide to recent research and its application*. London, UK: Cassell.

Olive, J. (1999). From fractions to rational numbers of arithmetic: A reorganization hypothesis. *Mathematical Thinking and Learning*, *1*(4), 279–314.

Programme for international Student Assessment (2003). Programme for international student assessment (PISA)—Country profiles. Retrieved from http://pisacountry.acer.edu.au/.

Resnick, L. B. (1987). Understanding algebra. In J. A. Sloboda & D. Rogers (Eds.), *Cognitive process in mathematics* (pp. 169–203). Oxford, UK: Clarendon.

Resnick, L. B. (2006). *The dilemma of mathematical intuition in learning*. Paper presented at the 30th conference of the international group for the psychology of mathematics education, Prague, Czech Republic. Q15

Sackur, C. (1995). *Blind calculators in algebra: Write false interviews*. Paper presented at the First European Research Conference on Mathematics Education (ERCME '95) Osnabruck, Germany. Q16

Schmittau, J. (2005). The development of algebraic thinking. *ZDM*, *37*(1), 16–22.

Schoenfeld, A. H., Smith, J., & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* Vol. 4, (pp. 55–175). Hillsdale, NJ: Lawrence Erlbaum.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1–36.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior*, *14*, 15–39.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification—The case of algebra. *Educational Studies in Mathematics*, *26*, 191–228.

Stacey, K., & MacGregor, M. (1997). Ideas about symbolism that students bring to algebra. *The Mathematics Teacher*, *90*(2), 110–113.

Stafylidou, S., & Vosniadou, S. (2004). Students' understanding of the numerical value of fractions: A conceptual change approach. *Learning and Instruction*, *14*, 503–518.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, *4*, 259–309.

Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research*. Dordrecht, The Netherlands: Kluwer Academic.

Tirosh, D., Even, R., & Robinson, N. (1998). Simplifying algebraic expressions: Teacher awareness and teaching approaches. *Educational studies in mathematics*, *35*, 51–64.

Vamvakoussi, X., & Vosniadou, S. (2007). How many numbers are there in a rational numbers interval? Constraints, synthetic models, and the effect of the number line. In S. Vosniadou, A. Baltas, & X. Vamvakoussi (Eds.), *Re-framing the conceptual change approach in learning and instruction* (pp. 265–282). Oxford, UK: Elsevier.

Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction*, *28*(2), 181–209.

Verschaffel, L., & Vosniadou, S. (Eds.). (2004). *The conceptual change approach to mathematics learning and teaching. Special Issue of Learning and Instruction* (Vol. 14). Q17

Vlassis, J. (2004). Making sense of the minus sign or becoming flexible in "negativity." *Learning and Instruction*, *14*, 469–484.

Vosniadou, S. (2003). Exploring the relationships between conceptual change and intentional learning. In G. M. Sinatra & P. R. Pintrich (Eds.), *Intentional conceptual change* (pp. 377–406). Mahwah, NJ: Lawrence Erlbaum.

Vosniadou, S., Vamvakoussi, X., & Skopeliti, E. (2008). The framework theory approach to conceptual change. In S. Vosniadou (Ed.), *Handbook of research on conceptual change* pp. 3–34. Mahwah, NJ: Lawrence Erlbaum.

Wagner, S. (1981). Conservation of equation and function under transformations of variables. *Journal for Research in Mathematics Education*, *12*, 107–118.

Warren, E., & Pierce, R. (2004). A review of research in learning and teaching algebra. In P. Perry, G. Anthony, & C. Diezmann (Eds.), *Research in mathematics education in Australia 2000–2003* (pp. 291–321). Flaxton, Australia: Post Pressed.