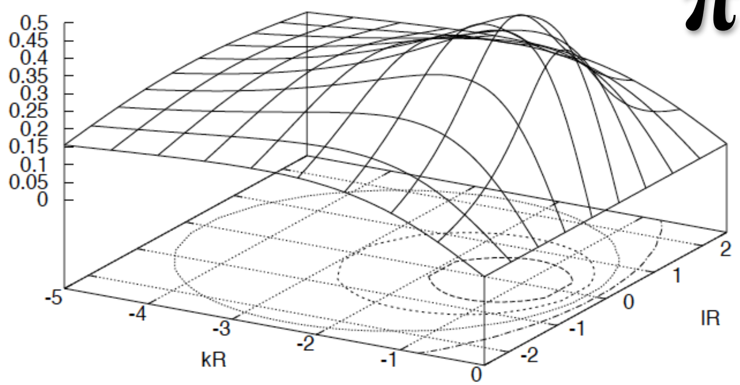
Ωκεάνια κύματα παρουσία περιστροφής



9. Rotating waves in the ocean

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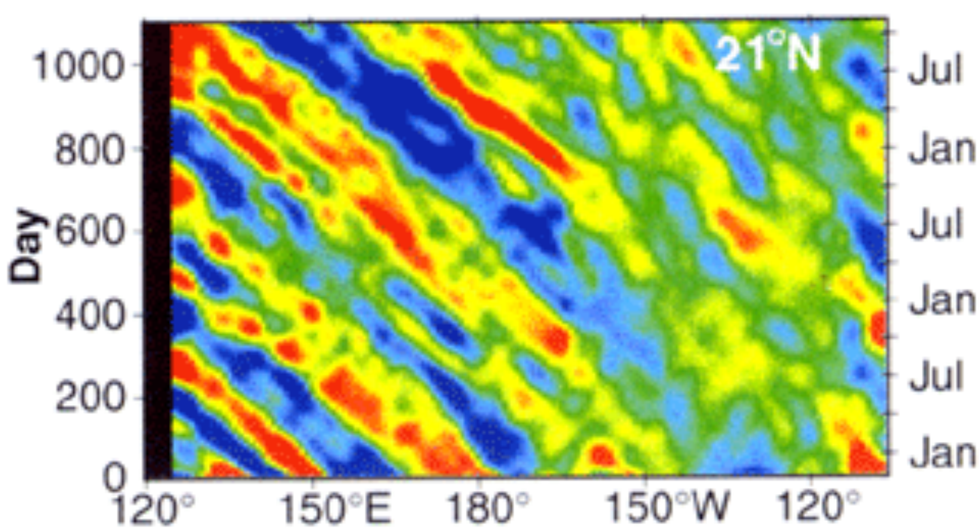


- **Wave solutions in a rotating ocean**
- **The *β*-effect**
- **Planetary waves**

Shallow water dynamics

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -g\frac{\partial \eta}{\partial x} + f\upsilon + A_H\frac{\partial^2 u}{\partial x^2} + A_H\frac{\partial^2 u}{\partial y^2} + A_V\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + w\frac{\partial \upsilon}{\partial z} = -g\frac{\partial \eta}{\partial y} - fu + A_H\frac{\partial^2 \upsilon}{\partial x^2} + A_H\frac{\partial^2 \upsilon}{\partial y^2} + A_V\frac{\partial^2 \upsilon}{\partial z^2}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) = 0$$

Approximations

$$R_o = \frac{U}{fL} << 1$$

$$R_{oT} = \frac{1}{fT} \approx 1$$

$$E_k = \frac{A_H}{fL^2} << 1$$

$f - plane$

Slow flow fields that evolve rapidly ←

$$U \neq C$$

Basic equations for ocean waves in the presence of rotation (f-plane)

$$(1)\quad \frac{\partial u}{\partial t} - f\upsilon = -g\frac{\partial \eta}{\partial x}$$

$$(2)\quad \frac{\partial \upsilon}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$(3)\quad \frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) = 0$$

$$\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) - f\left(\frac{\partial \upsilon}{\partial x} - \frac{\partial u}{\partial y}\right) = -g\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right)$$

using (3)

$$\frac{\partial}{\partial t}\left(-\frac{1}{H}\frac{\partial \eta}{\partial t}\right) + f\left(\frac{\partial \upsilon}{\partial x} - \frac{\partial u}{\partial y}\right) = g\nabla_h^2\eta \quad (4)$$

$$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \upsilon}{\partial x} - \frac{\partial u}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) = 0 \Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial \upsilon}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{f}{H}\frac{\partial \eta}{\partial t} \quad (5)$$

$$\frac{\partial}{\partial t}(4)$$

$$\frac{\partial}{\partial t}\frac{1}{H}\left(\frac{\partial^2\eta}{\partial t^2}\right)+f\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=g\frac{\partial}{\partial t}\nabla_h^2\eta$$

substituting (5)

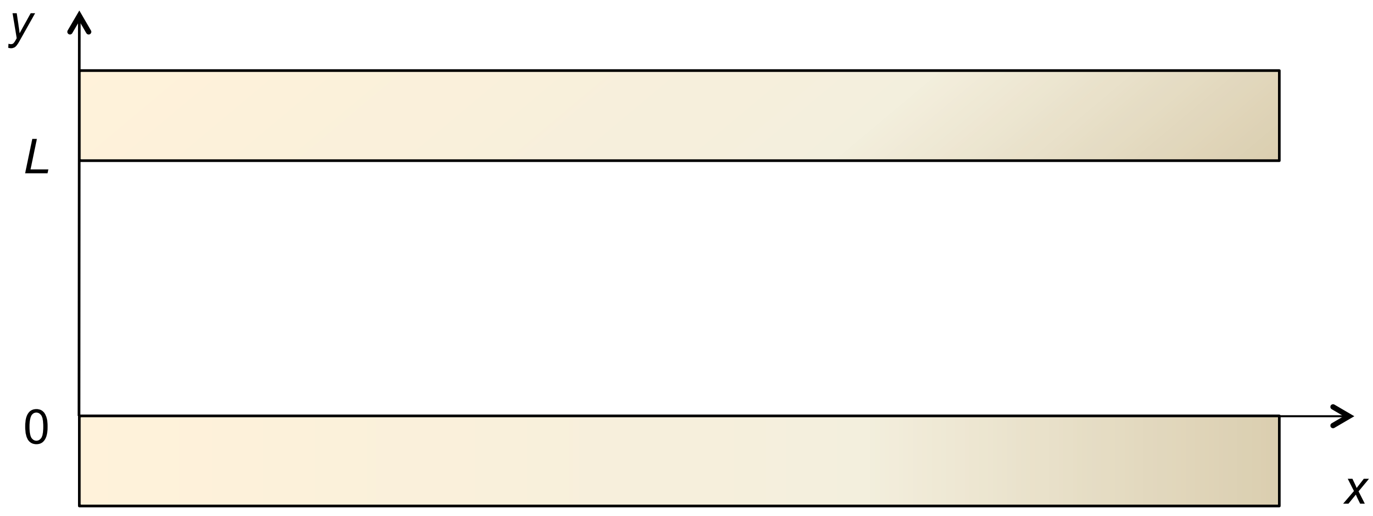
$$\frac{\partial}{\partial t}\left[\frac{\partial^2}{\partial t^2}+f^2\right]\eta=\underbrace{gH}_{C=\sqrt{gH}}\frac{\partial}{\partial t}\nabla_h^2\eta$$

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial t}\left[\frac{\partial^2}{\partial t^2}+f^2\right]\eta=C^2\frac{\partial}{\partial t}\nabla^2\eta} \quad \text{(A)}$$

$$\frac{\partial}{\partial t}(2)\rightarrow\frac{\partial^2 v}{\partial t^2}+f\frac{\partial u}{\partial t}=-g\frac{\partial^2\eta}{\partial t\partial y};\ \text{using (1)}\rightarrow\frac{\partial^2 v}{\partial t^2}+f\left(-g\frac{\partial\eta}{\partial x}+fv\right)=-g\frac{\partial^2\eta}{\partial t\partial y}$$

$$\Rightarrow\boxed{\frac{\partial^2 v}{\partial t^2}+f^2v=-g\left(\frac{\partial^2\eta}{\partial t\partial y}-f\frac{\partial\eta}{\partial x}\right)} \quad \text{(B)}$$

We can simplify the solution of the problem for wave propagation in a channel (bounded solutions).



where $\upsilon=0$ at $0,L$

We are seeking solutions of the form:

$$\eta = A(y)e^{i(kx-\omega t)}$$

Substituting in equation (A):

$$\frac{d^2A}{dy^2} + \left[\frac{\omega^2 - f^2}{C^2} - k^2\right]A = 0 \quad \text{or} \quad \frac{d^2A}{dy^2} + a^2 A = 0 \quad \text{for } a^2 = \frac{\omega^2 - f^2}{C^2} - k^2 \qquad (C_1)$$

Using the boundary conditions (at the channel walls) in (B)

$$\frac{\partial^2 \eta}{\partial t \partial y} - f\frac{\partial \eta}{\partial x} = 0 \quad \text{at} \quad y = 0, L \quad \text{or} \quad \frac{dA}{dy} + \frac{fk}{\omega} A = 0 \quad \text{at} \quad y = 0, L \qquad (C_2)$$

Substituting in (C_2) $A(y) = b_1 \sin(ay) + b_2 \cos(ay)$

at $y = 0$ $\quad b_1 a + \dfrac{fkb_2}{\omega} = 0$

at $y = L$ $\quad b_1 a\cos(aL) - b_2 a\sin(aL) + \dfrac{fkb_1}{\omega}\sin(aL) + \dfrac{fkb_2}{\omega}\cos(aL) = 0$

$$\det\begin{vmatrix} a & \dfrac{fk}{\omega} \\ a\cos(aL) + \dfrac{fk}{\omega}\sin(aL) & a\sin(aL) + \dfrac{fk}{\omega}\cos(aL) \end{vmatrix} = 0$$

The dispersion relation becomes:

$$\left(\omega^2 - f^2\right)^2 \cdot \left(\omega^2 - C^2 k^2\right) \cdot \sin(aL) = 0$$

This dispersion relation has three solutions

$$\omega = f$$ → ***Inertial oscillations***

$$\sin(aL) = 0 \Rightarrow a = \frac{n\pi}{L} \quad \text{where } n = 1,2,3,\ldots$$

$$\frac{n^2\pi^2}{L^2} = \frac{\omega^2 - f^2}{C^2} - k^2$$

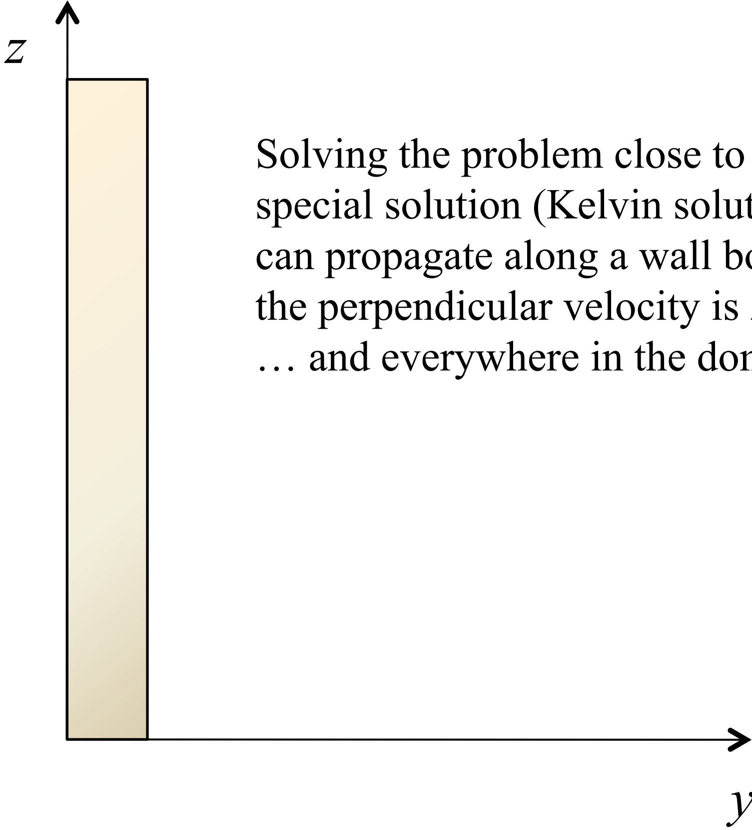
$$\Rightarrow$$

$$\omega = \pm\sqrt{f^2 + C^2\left(k^2 + \frac{n^2\pi^2}{L^2}\right)}$$ → ***Poincare waves***

$$\omega^2 = C^2k^2$$

$$\Rightarrow$$

$$\omega = \pm Ck$$ → ***Kelvin waves***



Solving the problem close to a wall has a special solution (Kelvin solution): The wave can propagate along a wall boundary (where the perpendicular velocity is zero on the wall … and everywhere in the domain)

$$\left.\begin{aligned} &(1) \qquad \frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \\ &(2) \qquad fu = -g\frac{\partial \eta}{\partial y} \\ &(3) \qquad \frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} = 0 \end{aligned}\right\}$$

Solutions: $$[u,\eta] = [\hat{u}(y),\hat{\eta}(y)]e^{i(kx-\omega t)}$$

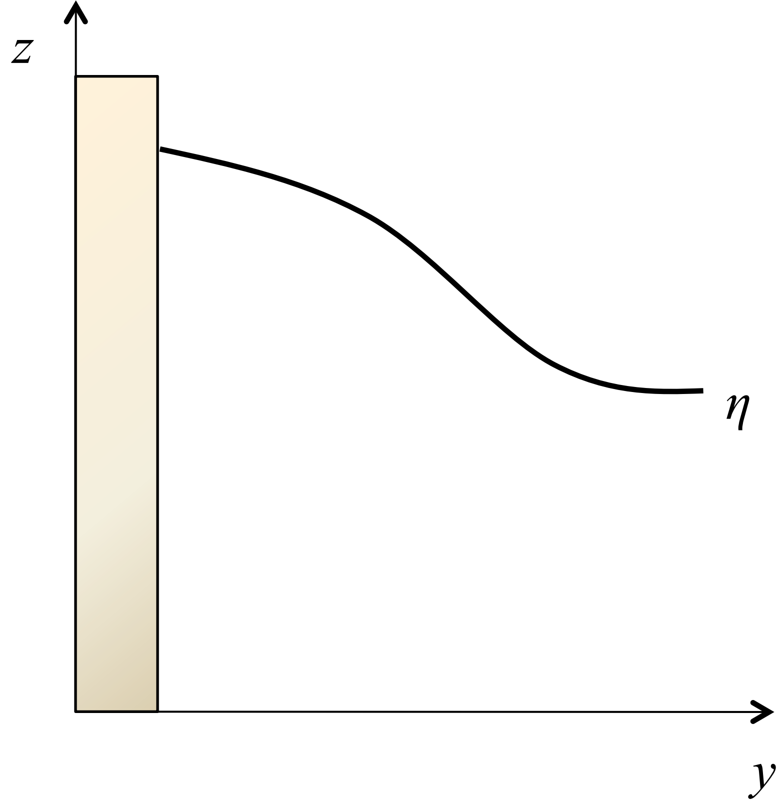
$$\left.\begin{aligned} &-i\omega\hat{u} - -igk\hat{\eta} \\ &f\hat{u} = -g\frac{d\hat{\eta}}{dy} \\ &-i\omega\hat{\eta} + iHk\hat{u} = 0 \end{aligned}\right\}$$

From (1) and (3):

$$\hat{\eta}\left(\omega^2 - gHk^2\right) = 0$$

$$\Rightarrow \omega = \pm k\sqrt{gH}$$

$$\boxed{c = \sqrt{gH}}$$



From equations (2) and (3):

$$\frac{d\hat{\eta}}{dy} + \frac{f}{c}\hat{\eta} = 0$$

Decaying solution: $\hat{\eta} = \eta_0 e^{-\frac{f}{c}y}$

$$\eta = \eta_0 e^{-\frac{f}{c}y} \cos k(x - ct)$$

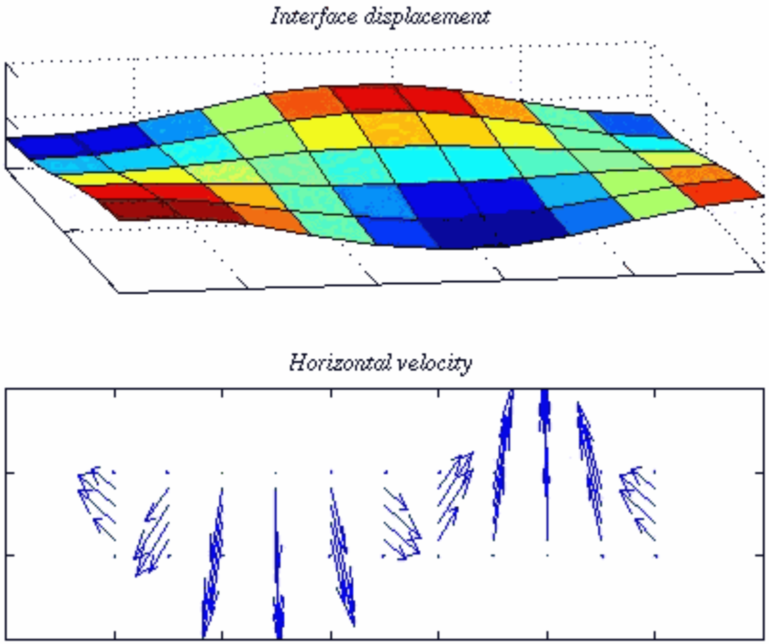
$$u = \eta_0 \sqrt{\frac{g}{H}} e^{-\frac{f}{c}y} \cos k(x - ct)$$

These are waves with maximum amplitude on the coast (having the "wall" on their right/left in the northern/southern hemisphere) and decaying exponentially away from the coast.

$$\frac{f}{C} = \frac{f}{\sqrt{gH}}$$

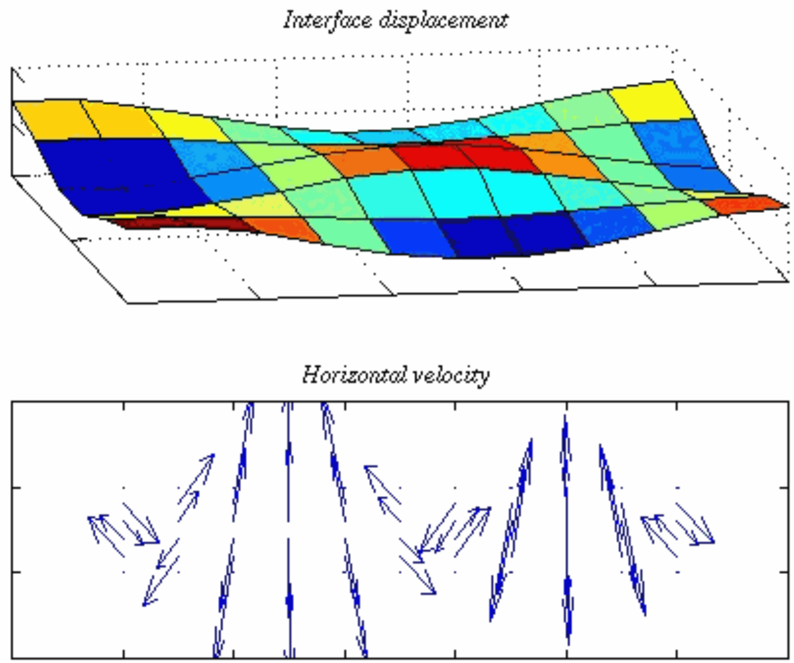
$$R_D = \frac{\sqrt{gH}}{f}$$ ***Rossby radius of deformation (external)***

Poincare waves in a channel

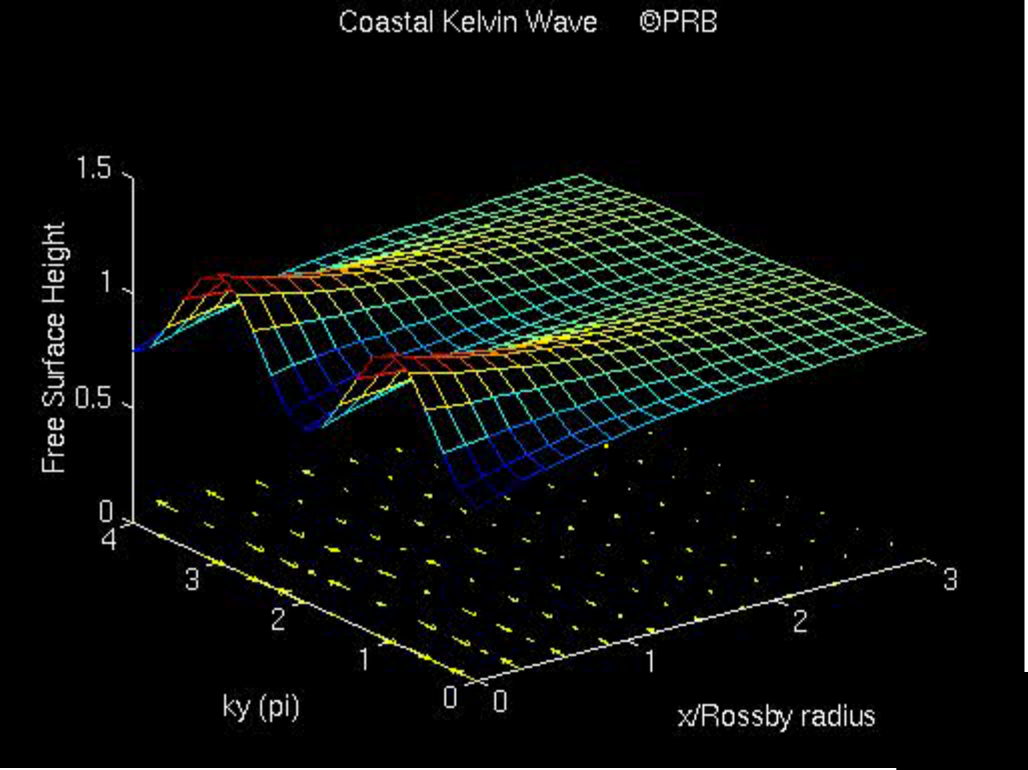


mode-1

mode-2

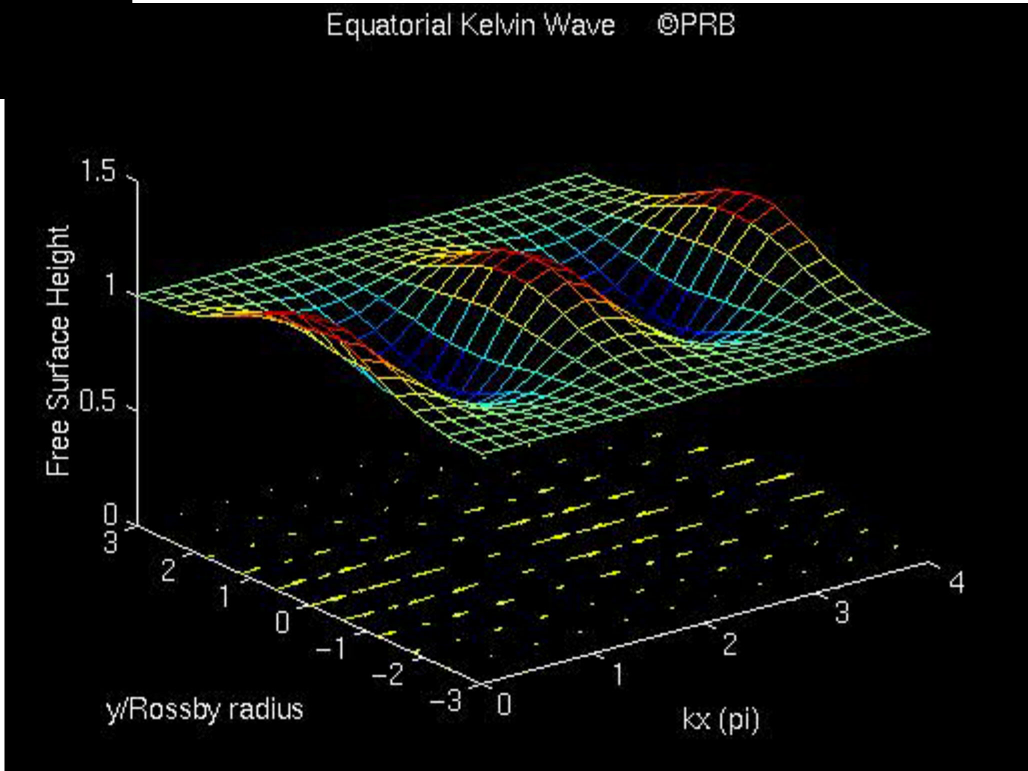


Kelvin waves



coastal

equatorial



Basic equations for ocean waves in the presence of rotation (f-plane)

$$R_{oT} = \frac{1}{fT} \approx 1$$

$$E_k = \frac{A_H}{fL^2} << 1$$

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t} - (f_0 + \beta y)v = -g\frac{\partial \eta}{\partial x} \\
&\frac{\partial v}{\partial t} + (f_0 + \beta y)u = -g\frac{\partial \eta}{\partial y} \\
&\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0
\end{aligned}\right\}$$

$\beta - plane$

$$f = f_0 + \beta y$$

At first order the system can approximated by geostrophy:

$$v \cong \frac{g}{f_0}\frac{\partial \eta}{\partial x}$$

$$u \cong -\frac{g}{f_0}\frac{\partial \eta}{\partial y}$$

Shallow Water Quasi-Geostrophy

Substituting in the equations of motion:

$$u = -\frac{g}{f_0}\frac{\partial \eta}{\partial y} - \frac{g}{f_0^{\,2}}\frac{\partial^2 \eta}{\partial x \partial t} + \frac{\beta g}{f_0^{\,2}} y \frac{\partial \eta}{\partial y}$$

$$\upsilon = \frac{g}{f_0}\frac{\partial \eta}{\partial x} - \frac{g}{f_0^{2}}\frac{\partial^2 \eta}{\partial y \partial t} - \frac{\beta g}{f_0^{2}} y \frac{\partial \eta}{\partial x}$$

geostrophic ***a-geostrophic***

Using the continuity equation: $\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) = 0$

$$\frac{\partial \eta}{\partial t} - R_D^{\,2}\frac{\partial}{\partial t}\nabla^2 \eta - \beta R_D^{\,2}\frac{\partial \eta}{\partial x} = 0$$

$$R_D = \frac{\sqrt{gH}}{f}$$

Wave solution:

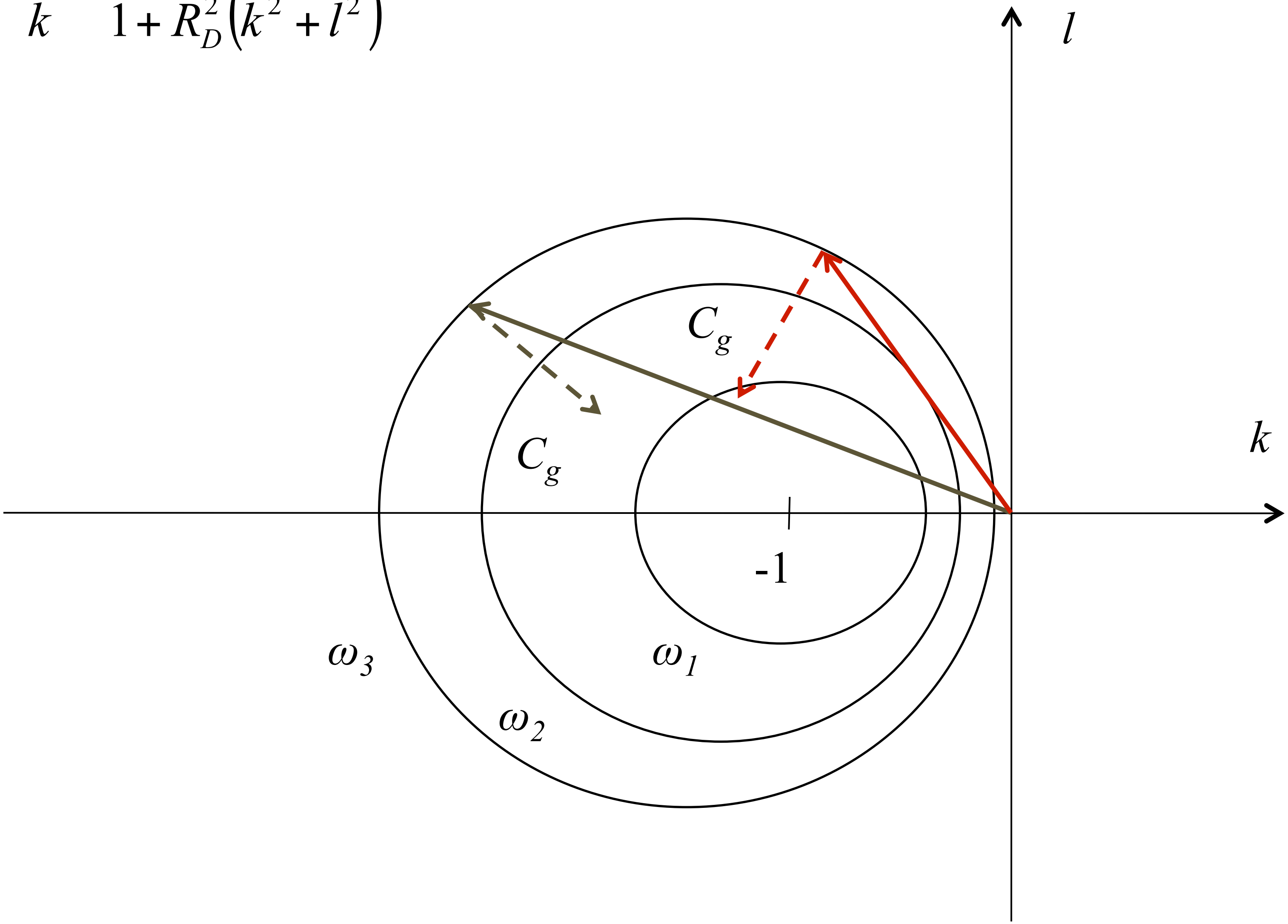
$$\eta \propto \cos(kx + ly - \omega t)$$

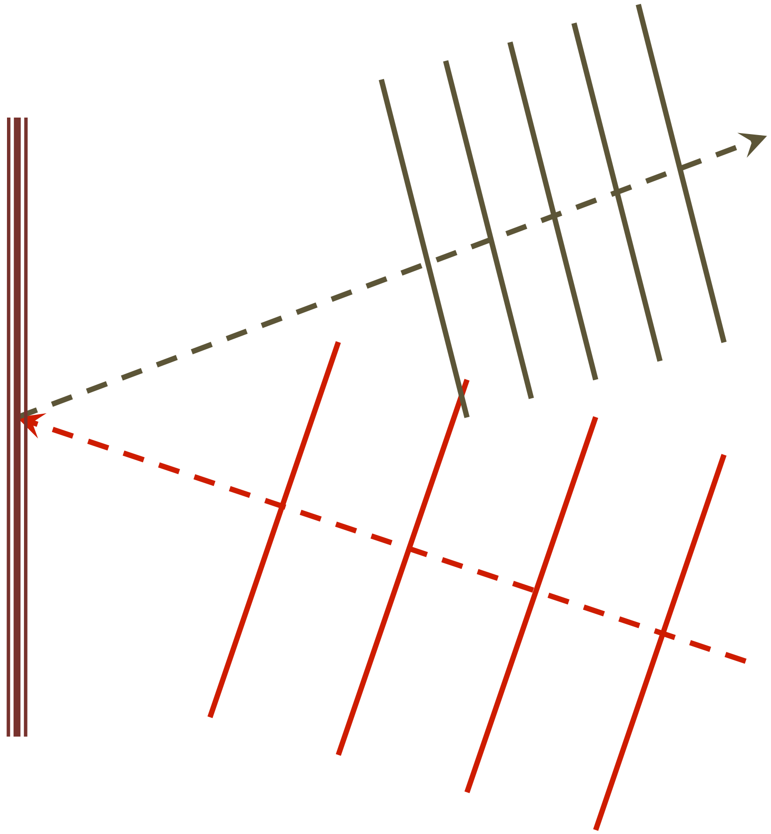
$$\omega = -\beta R_D^{\,2}\frac{k}{1 + R_D^{\,2}\left(k^2 + l^2\right)}$$

At the limit $\omega = \beta = 0 \rightarrow$ GEOSTROPHY

Dispersion diagram

$$C_x \equiv \frac{\omega}{k} = \frac{-\beta R_D^2}{1 + R_D^2\left(k^2 + l^2\right)}$$





Wavepacket reflection on the western boundary

$$\omega = -\frac{\beta k}{\frac{1}{R^2} + k^2}$$

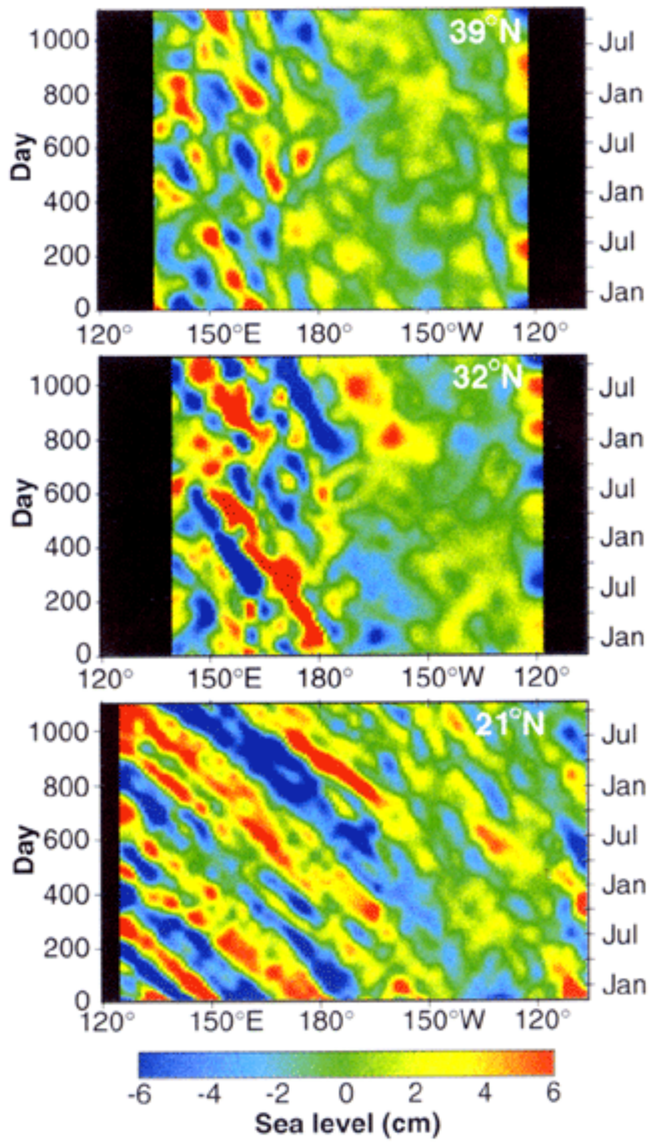
Short wavelengths: $k >> \frac{1}{R}$

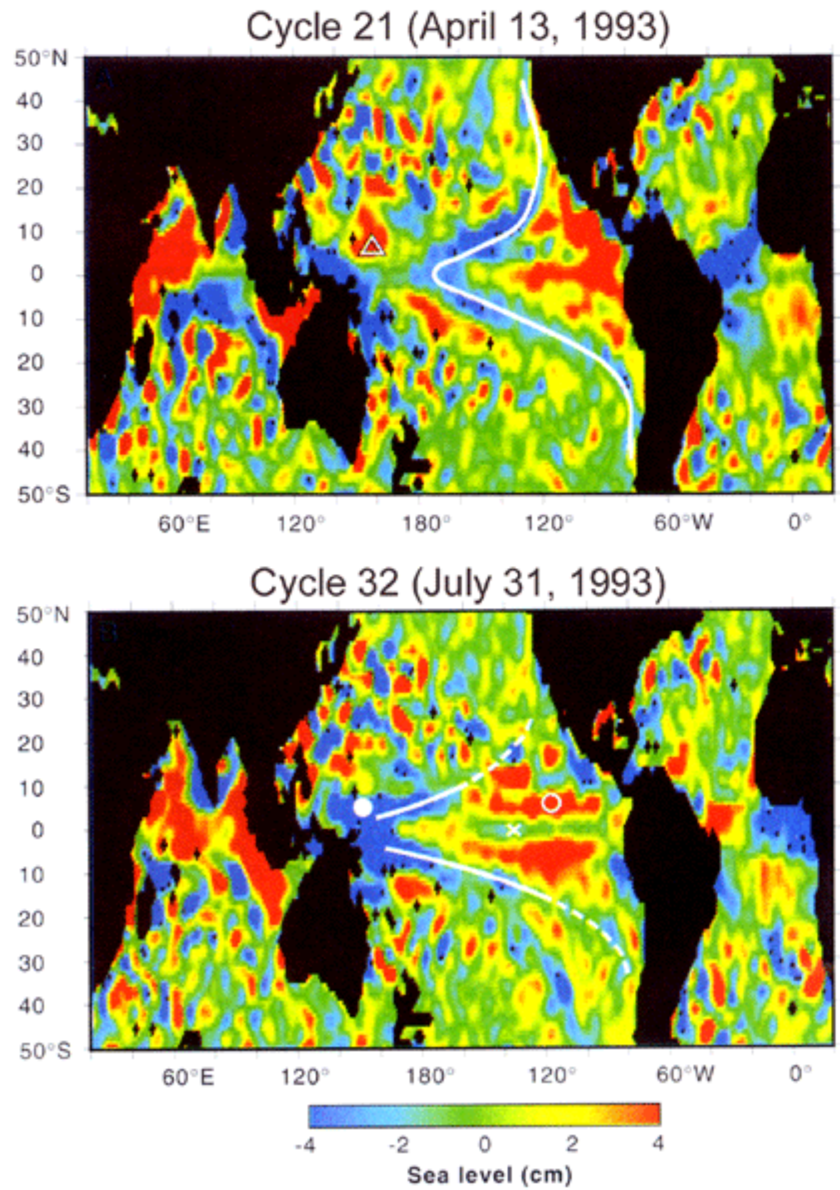
$$\rightarrow \omega \approx -\frac{\beta}{k} \quad \rightarrow \quad c_g = \frac{\beta}{k^2}$$

Long wavelengths: $k << \frac{1}{R}$

$$\rightarrow \omega \approx -\beta k R^2 \quad \rightarrow \quad c_g = -\beta R^2$$

$$c_g(L) > c_g(S)$$

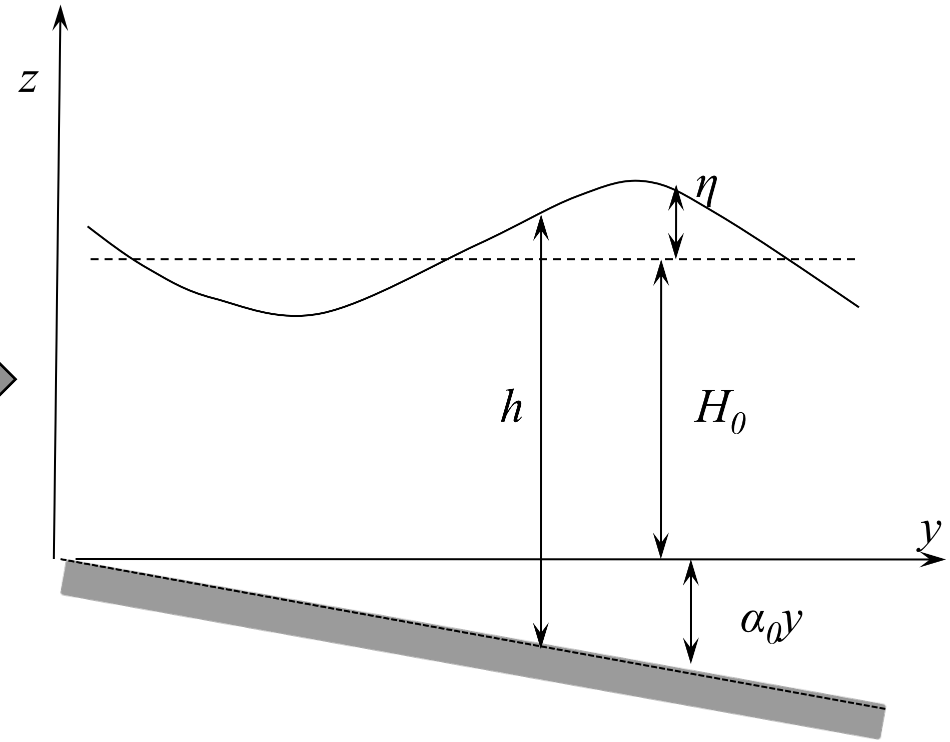

39°N
32°N
21°N
Day
0
200
400
600
800
1000
120°
150°E
180°
150°W
120°
Jan
Jul
1993
1994
1995
-6
-4
-2
0
2
4
6
Sea level (cm)


Cycle 21 (April 13, 1993)
Cycle 32 (July 31, 1993)
50°N
40
30
20
10
0
50°S
60°E
120°
180°
120°
60°W
0°
-4
-2
0
2
4
Sea level (cm)

Topographic waves (f-plane)

$$H = H_0 + a_0 y; \quad a = \frac{a_0 L}{H_0} << 1$$

$$h(x,y,t) = H_0 + ay + \eta(x,y,t)$$



The continuity equation becomes:

$$\frac{\partial \eta}{\partial t} + \left(H_0 + a_0 y + \eta\right)\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) + a_0 \upsilon = 0$$

$$a_0 y, \eta << H_0$$

$$\frac{\partial \eta}{\partial t} + H_0\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) + a_0 \upsilon = 0$$

Basic equations for oceanic waves in the presence of rotation (f-plane) and topographic influence

$$\frac{\partial u}{\partial t} - f\upsilon = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial \upsilon}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H_0\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) + a_0\upsilon = 0$$

Approximation (geostrophy):

$$\upsilon \cong \frac{g}{f}\frac{\partial \eta}{\partial x}$$

$$u \cong -\frac{g}{f}\frac{\partial \eta}{\partial y}$$

$$u = -\frac{g}{f}\frac{\partial \eta}{\partial y} - \frac{g}{f^2}\frac{\partial^2 \eta}{\partial x \partial t}$$

$$\upsilon = \frac{g}{f}\frac{\partial \eta}{\partial x} - \frac{g}{f^2}\frac{\partial^2 \eta}{\partial y \partial t}$$

$$\frac{\partial \eta}{\partial t} - R^2\frac{\partial}{\partial t}\nabla^2\eta - \frac{a_0 g}{f}\frac{\partial \eta}{\partial x} = 0$$

$$R = \frac{\sqrt{gH_0}}{f}$$

$$\frac{\partial \eta}{\partial t} - R^2 \frac{\partial}{\partial t}\nabla^2\eta - \frac{a_0 g}{f}\frac{\partial \eta}{\partial x} = 0$$

κυματική λύση:

$$\eta \propto \cos(kx + ly - \omega t)$$

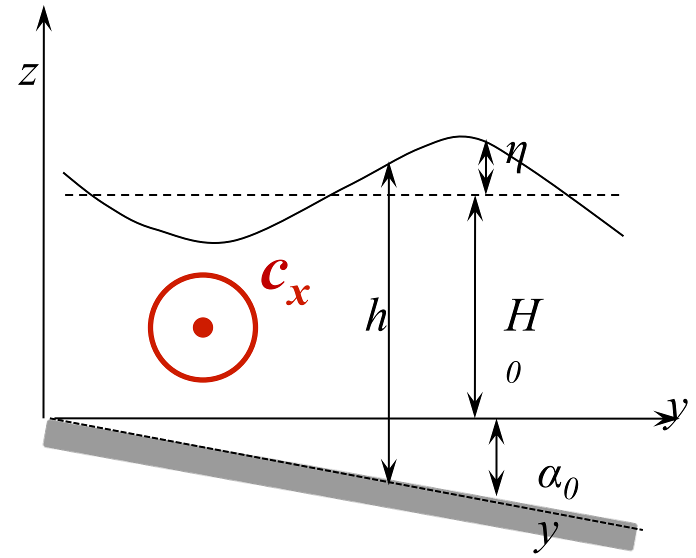
$$\omega = \frac{a_0 g}{f}\frac{k}{1 + R^2\left(k^2 + l^2\right)}$$

$$c_x = \frac{a_0 g}{f}\frac{1}{1 + R^2\left(k^2 + l^2\right)}$$

<u>**Βόρειο ημισφαίριο**</u>: Μικρά βάθη στα δεξιά

<u>**Νότιο ημισφαίριο**</u>: Μικρά βάθη στα αριστερά

$$c_x(\max) = \frac{a_0 g}{f}$$

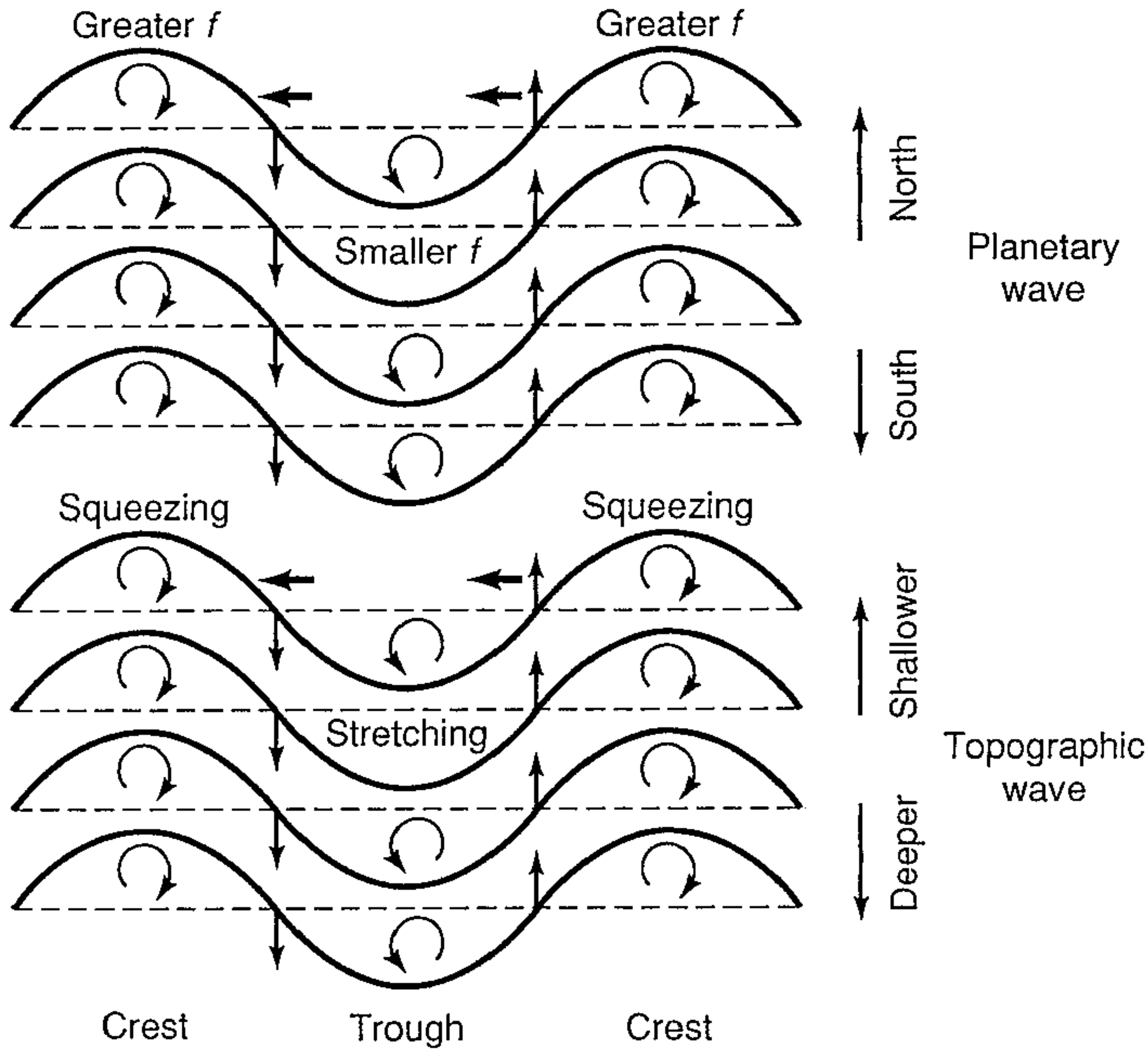


$$f = f_0 + \beta y$$

Αναλογίες και μηχανισμοί πλανητικών-τοπογραφικών κυμάτων

$$\frac{d}{dt}\left(\frac{f_0 + \beta y + \zeta}{H_0 + a_0 y}\right)$$

$$H = H_0 + a_0 y$$



OBSERVING OCEAN WAVES