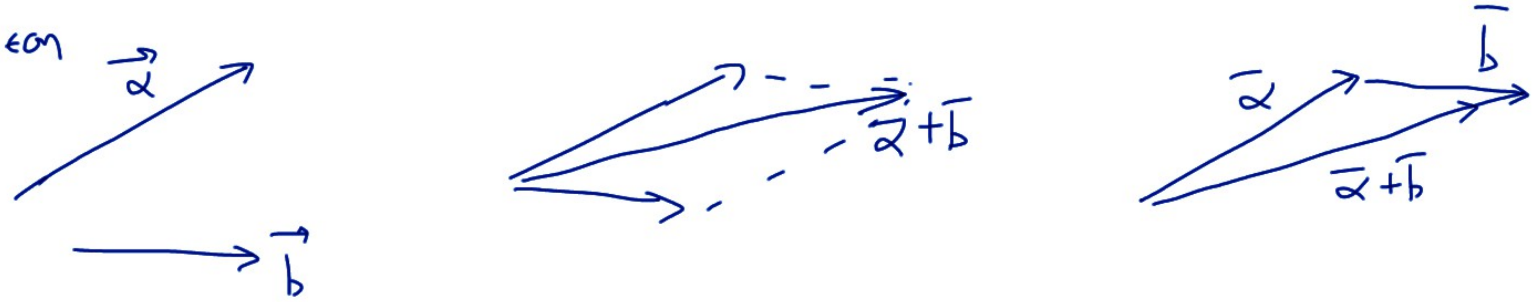
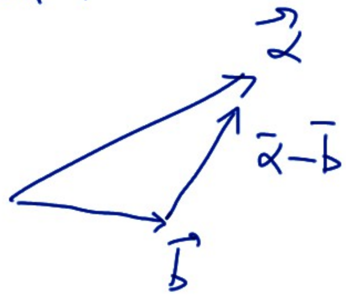
Πράξεις διανυσμάτων

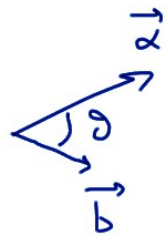
Πρόσθεση



Αφαίρεση :



Εσωτερικό γινόμενο $\quad \vec{\alpha}\cdot\bar{b} = |\vec{\alpha}|\,|\bar{b}|\cos\theta$



$$\bar{\alpha}\cdot\bar{b} = \bar{b}\cdot\bar{\alpha}$$

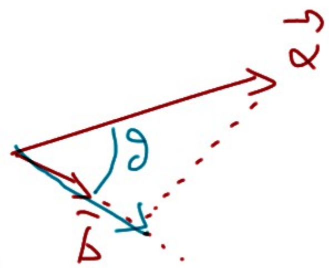
$$\bar{\alpha}\cdot(\bar{b}+\bar{c}) = \bar{\alpha}\cdot\bar{b} + \bar{\alpha}\cdot\bar{c}$$

$$(\bar{\alpha}\cdot\bar{b})\bar{c} \neq \bar{\alpha}\,(\bar{b}\cdot\bar{c})$$

$$\bar{\alpha}\cdot\bar{b} = 0 \Leftrightarrow \bar{\alpha}\perp\bar{b}$$

$$|\bar{\alpha}| = \sqrt{\bar{\alpha}\cdot\bar{\alpha}}$$

Προβολή του $\bar{\alpha}$ πάνω στο $\bar{b}$



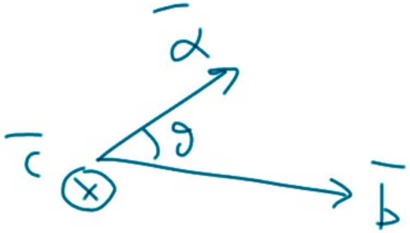
$$|\bar{\alpha}|\cos\theta\ \hat{b} = (\bar{\alpha}\cdot\hat{b})\,\hat{b} = \frac{(\bar{\alpha}\cdot\bar{b})\,\bar{b}}{|\bar{b}|^2}$$

$\bar{b}/|\bar{b}|$

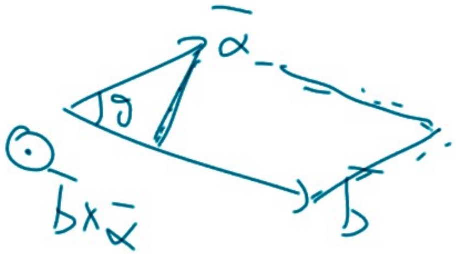
Εξωτερικό γινόμενο

$$\bar{\alpha} \times \bar{b} = \bar{c}$$

$$|\bar{c}| = |\bar{\alpha}||\bar{b}|\sin\theta = \text{εμβαδόν παρ/μου}$$



$$\bar{b} \times \bar{\alpha} = -\bar{\alpha} \times \bar{b}$$



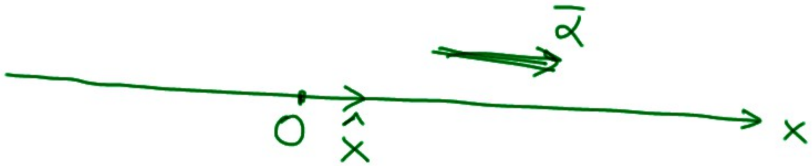
$$\bar{\alpha} // \bar{b} \Leftrightarrow \bar{\alpha} \times \bar{b} = 0$$

Μικτά γινόμενα $(\bar{\alpha} \times \bar{b}) \cdot \bar{c} = \bar{\alpha} \cdot (\bar{b} \times \bar{c}) = \bar{b} \cdot (\bar{c} \times \bar{\alpha})$

$$\bar{\alpha} \times (\bar{b} \times \bar{c}) = (\bar{\alpha} \cdot \bar{c})\,\bar{b} - (\bar{\alpha} \cdot \bar{b})\,\bar{c}$$

$$(\bar{\alpha} \times \bar{b}) \times \bar{c} = (\bar{\alpha} \cdot \bar{c})\,\bar{b} - (\bar{b} \cdot \bar{c})\,\bar{\alpha}$$

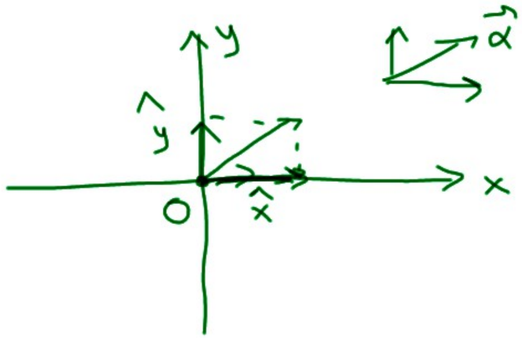
1D



$$\bar{\alpha} = \alpha_x \hat{x}$$

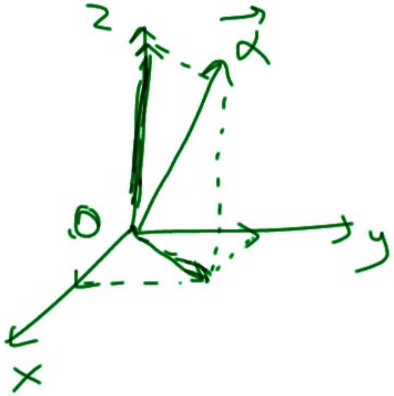
αλγεβρική τιμή

2D



$$\bar{\alpha} = \underbrace{(\bar{\alpha}\cdot\hat{x})}_{\alpha_x}\hat{x} + \underbrace{(\bar{\alpha}\cdot\hat{y})}_{\alpha_y}\hat{y}$$

3D



$$\hat{x} \times \hat{y} = \hat{z}$$

$$\boxed{\bar{\alpha} = (\bar{\alpha}\cdot\hat{x})\hat{x} + (\bar{\alpha}\cdot\hat{y})\hat{y} + (\bar{\alpha}\cdot\hat{z})\hat{z}}$$

σχέση πληρότητας

$$\bar{\alpha} = \underbrace{\alpha_x \hat{x}} + \alpha_y \hat{y} + \alpha_z \hat{z} \quad , \quad \bar{b} = \underbrace{b_x \hat{x}} + \underbrace{b_y \hat{y}} + b_z \hat{z}$$

$$\bar{\alpha} \cdot \bar{b} = \alpha_x b_x + \alpha_y b_y + \alpha_z b_z \quad , \quad |\bar{\alpha}| = \sqrt{\bar{\alpha} \cdot \bar{\alpha}} = \sqrt{\alpha_x^2 + \alpha_y^2 + \alpha_z^2}$$

$$\bar{\alpha} \times \bar{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \alpha_x & \alpha_y & \alpha_z \\ b_x & b_y & b_z \end{vmatrix} \qquad (\bar{\alpha} \times \bar{b}) \cdot \bar{c} = \begin{vmatrix} \alpha_x & \alpha_y & \alpha_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$\bar{\alpha}(t) \qquad \frac{d\bar{\alpha}}{dt} = \lim_{\Delta t \to 0} \frac{\bar{\alpha}(t + \Delta t) - \bar{\alpha}(t)}{\Delta t}$$

$$\frac{d}{dt}(\bar{\alpha} \cdot \bar{b}) = \frac{d\bar{\alpha}}{dt} \cdot \bar{b} + \bar{\alpha} \cdot \frac{d\bar{b}}{dt}$$

$$\frac{d}{dt}(\bar{\alpha} \times \bar{b}) = \frac{d\bar{\alpha}}{dt} \times \bar{b} + \bar{\alpha} \times \frac{d\bar{b}}{dt}$$

$$\frac{d}{dt}(\lambda \bar{\alpha}) = \frac{d\lambda}{dt} \bar{\alpha} + \lambda \frac{d\bar{\alpha}}{dt}$$

Δείξτε ότι $\bar{\alpha} = \underbrace{(\bar{\alpha}\cdot\hat{b})\,\hat{b}}_{\text{παράλληλη συνιστώσα στο } \hat{b}} + \underbrace{(\hat{b}\times\bar{\alpha})\times\hat{b}}_{\text{κάθετη συνιστώσα στο } \hat{b}}$

Απόδειξη: $(\hat{b}\times\bar{\alpha})\times\hat{b} = \underbrace{(\hat{b}\cdot\hat{b})}_{1}\,\bar{\alpha} - (\bar{\alpha}\cdot\hat{b})\,\hat{b}$

Δείξτε ότι η παράγωγος ενός μοναδιαίου $\hat{\varepsilon}$ είναι κάθετη στο $\hat{\varepsilon}$.

Απόδειξη: $\cancel{\frac{d\hat{\varepsilon}}{dt}\cdot\hat{\varepsilon}=0}$ Παραγωγίζω την $|\hat{\varepsilon}|^2=1 \Leftrightarrow \hat{\varepsilon}\cdot\hat{\varepsilon}=1$

$$\frac{d\hat{\varepsilon}}{dt}\cdot\hat{\varepsilon} + \hat{\varepsilon}\cdot\frac{d\hat{\varepsilon}}{dt} = 0 \Leftrightarrow \frac{d\hat{\varepsilon}}{dt}\cdot\hat{\varepsilon} = 0 \quad \text{δηλ.} \quad \frac{d\hat{\varepsilon}}{dt} \perp \hat{\varepsilon}$$

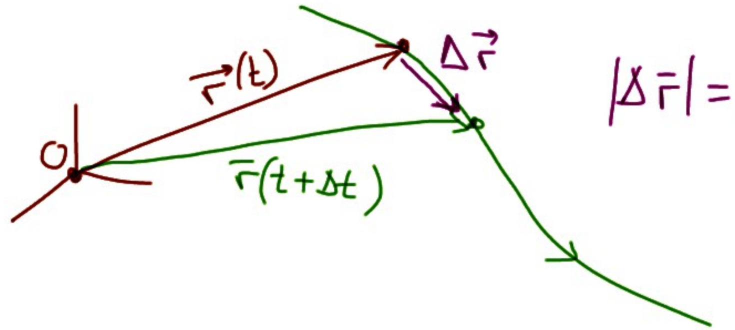
Παραγωγίζω την $\hat{\varepsilon}\times\hat{\varepsilon}=0$ και έχω $2\frac{d\hat{\varepsilon}}{dt}\times\hat{\varepsilon}=0$ δηλ. $\frac{d\hat{\varepsilon}}{dt} \parallel \hat{\varepsilon}$

Που είναι το λάθος;

Το σωστό: $\frac{d}{dt}(\hat{\varepsilon}\times\hat{\varepsilon})=0 \Leftrightarrow \frac{d\hat{\varepsilon}}{dt}\times\hat{\varepsilon}+\hat{\varepsilon}\times\frac{d\hat{\varepsilon}}{dt}=0$ ισχύει (δεν δίνει)

Κινηματική

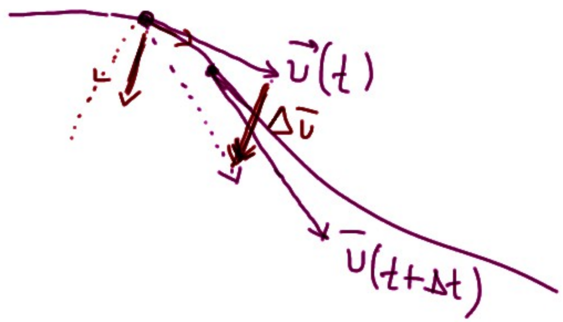
$$\bar{r} = \bar{r}(t)$$



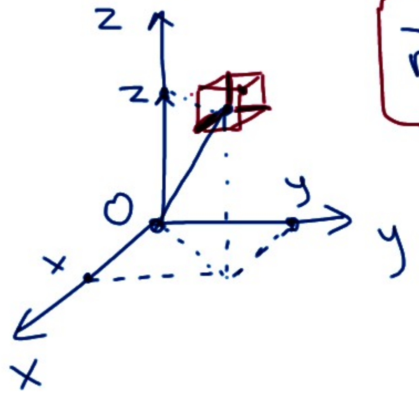
$$|\Delta \bar{r}| = \Delta s$$

$$\bar{v} = \frac{d\bar{r}}{dt} = \dot{\bar{r}}$$

$$\bar{\alpha} = \frac{d\bar{v}}{dt} = \dot{\bar{v}} \quad ή \quad \bar{\alpha} = \frac{d^2\bar{r}}{dt^2} = \ddot{\bar{r}}$$



= Καρτεσιανές συντεταγμένες $x(t),\ y(t),\ z(t)$



$$\boxed{\bar{r} = x\hat{x} + y\hat{y} + z\hat{z}} \quad (*)$$

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$ds = |d\bar{r}| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

$$d\bar{S} = dx\,dy\,\hat{z} \quad \text{κοκ} \qquad dV = dx\,dy\,dz$$

$$\bar{u} = \frac{d\bar{r}}{dt} = \frac{dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}}{dt} = \boxed{\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} = \vec{u}}$$

(το ίδιο παραγωγίζοντας την (*))

$$\bar{\alpha} = \dot{\bar{u}} = \frac{d}{dt}\left(\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}\right) = \boxed{\ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z} = \vec{\alpha}}$$

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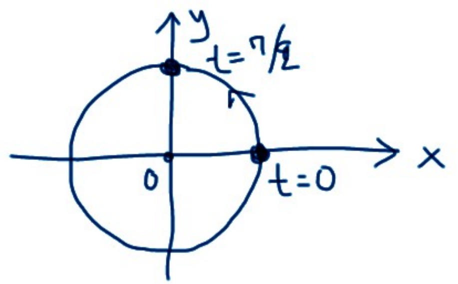
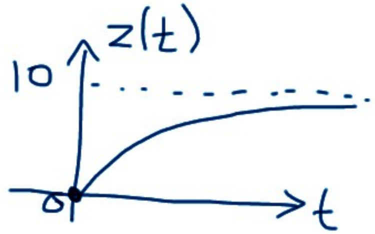
$$\bar{r} = \underbrace{\cos t}_{x(t)}\,\hat{x} + \underbrace{\sin t}_{y(t)}\,\hat{y} + \underbrace{10\left(1-e^{-t/10}\right)}_{z(t)}\,\hat{z}$$

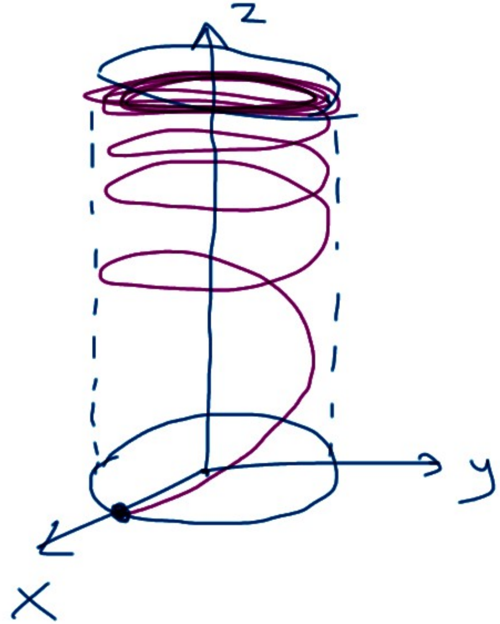
(α) Προβολή τροχιάς στο xy επίπεδο.

$$x^2+y^2=1$$





(β) Τρισδιάστατη τροχιά;

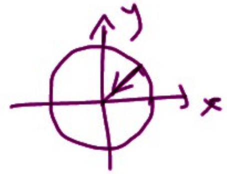


(γ) $\bar{v} = \dot{\bar{r}} = \underbrace{-\sin t}_{v_x = \dot{x}}\, \hat{x} + \underbrace{\cos t}_{v_y}\, \hat{y} + \underbrace{e^{-t/10}}_{v_z}\, \hat{z}$

$\left(\bar{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} \right)$

$\bar{\alpha} = \dot{v}_x \hat{x} + \dot{v}_y \hat{y} + \dot{v}_z \hat{z}$

ή $\ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$ $= \underbrace{-\cos t\, \hat{x} - \sin t\, \hat{y}} - \frac{1}{10} e^{-t/10} \hat{z}$



$\alpha_\kappa = \frac{v_{xy}^2}{R}$

(δ) $\hat{z}$ συνιστώσα στροφορμής

$\vec{L} = \bar{r} \times m\bar{v} = m \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}$, $L_z = m \begin{vmatrix} x & y \\ \dot{x} & \dot{y} \end{vmatrix} = m(x\dot{y} - \dot{x}y) =$

$= m[\cos t \cdot \cos t - (-\sin t)\cdot \sin t] = m = \sigma\tau\alpha\vartheta.$ $\dot{L}_z = (\bar{r} \times m\bar{\alpha}) \cdot \hat{z} = \ldots = 0$

Β' τρόπος : $\dot{L}_z = \dot{\vec{L}} \cdot \hat{z}$, $\dot{\vec{L}} = \cancel{\dot{\bar{r}} \times m\bar{v}} + \bar{r} \times m\dot{\bar{v}} = \bar{r} \times m\bar{\alpha}$, (ε) $t = ;$ $\bar{\alpha} \perp \bar{r}$

$\bar{\alpha} \cdot \bar{r} = 0 \Leftrightarrow -1 - e^{-t/10}\left(1 - e^{-t/10}\right) = 0$ τριώνυμο ως προς $e^{-t/10}$

$\left(e^{-t/10}\right)^2 - \left(e^{-t/10}\right) - 1 = 0$ λύσεις $e^{-t/10} = \dfrac{1 \pm \sqrt{5}}{2}$

Η λύση με το $-$ απορρίπτεται (διότι $e^{-t/10} > 0$)

Άρα $e^{-t/10} = \dfrac{1+\sqrt{5}}{2} \Leftrightarrow -\dfrac{t}{10} = \ln \dfrac{1+\sqrt{5}}{2} \Leftrightarrow t = -10 \ln \dfrac{1+\sqrt{5}}{2}$

$t < 0$ αλλά την δέχομαι

(στ) $\bar{\alpha} = \alpha_x(x)\,\hat{x} + \alpha_y(y)\,\hat{y} + \alpha_z(z)\,\hat{z}$

$\alpha_x = -\cos t = -x$, $\alpha_y = -\sin t = -y$, $\alpha_z = -\dfrac{1}{10} e^{-t/10} = \dfrac{z-10}{100}$

$z = 10\left(1 - e^{-t/10}\right) \Leftrightarrow e^{-t/10} = 1 - \dfrac{z}{10}$

(J) $\bar{\alpha} = \bar{w} \times \bar{v} - k v_z \hat{z}$ όπου $\bar{w} // \hat{z}$ σταθ. , $k =$ σταθ

Πρέπει να ισχύει:

$$-\cos t\,\hat{x} - \sin t\,\hat{y} - \frac{1}{10} e^{-t/10} \hat{z} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & w \\ -\sin t & \cos t & e^{-t/10} \end{vmatrix} - k e^{-t/10} \hat{z} \iff$$

$$\iff \begin{cases} \hat{x}: -\cos t = -w \cos t & \text{δηλ. } w = 1 \quad (\bar{w} = \hat{z}) \\ \hat{y}: -\sin t = -w \sin t & \checkmark \\ \hat{z}: -\frac{1}{10} e^{-t/10} = -k e^{-t/10} & \text{δηλ. } k = 1 \end{cases}$$

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$$\bar v_B = \bar v_A \times \hat z \Leftrightarrow \frac{d\bar r_B}{\cancel{dt}} = \frac{d\bar r_A}{\cancel{dt}} \times \hat z \Leftrightarrow \underbrace{\int d\bar r_B}_{\bar r_B} = \underbrace{\int d\bar r_A}_{\bar r_A + \sigma\tau\alpha\theta} \times \hat z$$

$$\bar r_B = \bar r_A \times \hat z + \bar C$$

Από αρχικές συνθήκες $R\hat x = 0 + \bar C \Leftrightarrow \bar C = R\hat x$.

Άρα σε κάθε χρόνο $\bar r_B = \bar r_A \times \hat z + R\hat x$

Την στιγμή της συνάντησης $\bar r_A = \bar r_B = \bar r_\Sigma$ οπότε $\bar r_\Sigma = \bar r_\Sigma \times \hat z + R\hat x$

$$\Leftrightarrow x_\Sigma \hat x + y_\Sigma \hat y = \begin{vmatrix} \hat x & \hat y & \hat z \\ x_\Sigma & y_\Sigma & z_\Sigma \\ 0 & 0 & 1 \end{vmatrix} + R\hat x \Leftrightarrow \left\{ \begin{array}{ll} \hat x: & x_\Sigma = y_\Sigma + R \\ \hat y: & y_\Sigma = -x_\Sigma \end{array} \right\} \Leftrightarrow \begin{array}{l} x_\Sigma = R/2 \\ y_\Sigma = -R/2 \end{array}$$