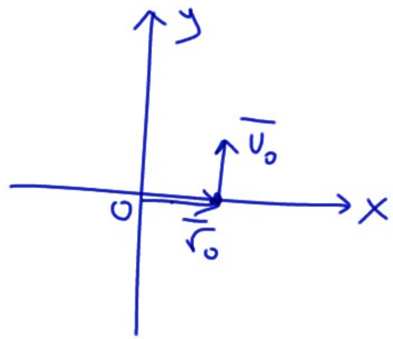
Θέμα 3 εξέταση 27/5/2013

$m=1$, $V=\dfrac{4}{r^6}-\dfrac{5}{r^4}$, $\bar{r}_0=\hat{x}$, $\bar{v}_0=\sqrt{2}\,\hat{y}$

(α) $E=\underbrace{\dfrac{mv^2}{2}}+V=\dfrac{m\dot{r}^2}{2}+V_{eff}$

$$\frac{m\dot{r}^2}{2}+\frac{m(r\dot{\varphi})^2}{2}=\frac{m\dot{r}^2}{2}+\frac{L^2}{2mr^2}$$



$$E=\frac{v_0^2}{2}+V(1)=0$$

$$\vec{L}=m\,\bar{r}_0\times\bar{v}_0=\hat{x}\times\sqrt{2}\,\hat{y}=\sqrt{2}\,\hat{z} \qquad L=m r_0 v_0=\sqrt{2}$$

$$V_{eff}=\frac{L^2}{2mr^2}+V=\frac{1}{r^2}+\frac{4}{r^6}-\frac{5}{r^4}$$

$$\left(E=\frac{m\dot{r}^2}{2}+V_{eff}=\cancel{\frac{m\dot{r}^2}{2}}\Big|_{t=0}+V_{eff}(1)=0\right)$$

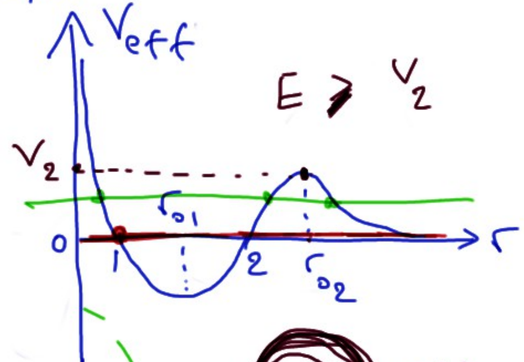
(β) κυκλικές τροχιές $\Rightarrow V'_{eff}(r) = -\dfrac{2}{r^3} + \dfrac{20}{r^5} - \dfrac{24}{r^7} = 0$

$$\Leftrightarrow r^4 - 10r^2 + 12 = 0 \Leftrightarrow r^2 = 5 \pm \sqrt{13} \Leftrightarrow r_{0_1} = \sqrt{5-\sqrt{13}}, \quad r_{0_2} = \sqrt{5+\sqrt{13}} .$$

(Αλλιώς: $\dfrac{mv^2}{r} = |F| \xrightarrow[v=\frac{L}{r}=\frac{\sqrt{2}}{r}]{L=\sqrt{2}} \dfrac{2}{r^3} = |F| \; \ldots$)

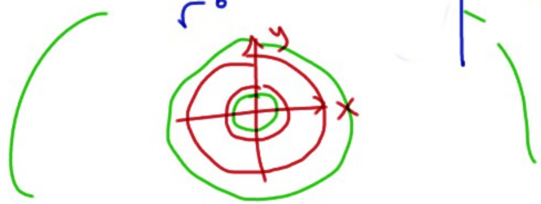
$$V'_{eff}(r) = \frac{-2}{r^7}\Big(\underbrace{r^4 - 10r^2 + 12}_{\text{αρνητικό για } r_{0_1} < r < r_{0_2}}\Big)$$

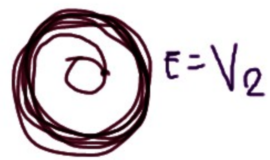
Το r_{0_1} ελάχιστο άρα ευσταθής η κυκλική τροχιά ενώ το r_{0_2} μέγιστο άρα ασταθής.



$$V_{eff} = \frac{1}{r^2} + \frac{4}{r^6} - \frac{5}{r^4} = \frac{r^4 - 5r^2 + 4}{r^6} = \frac{(r^2-1)(r^2-4)}{r^6}$$

(γ) $V_{eff} \leq E \Leftrightarrow V_{eff} \leq 0 \Leftrightarrow 1 \leq r \leq 2$





(Ε) $u'' + u = -\dfrac{mF}{L^2 u^2}$, $F = -\dfrac{dV}{dr} = 24u^7 - 20u^5$, $L = \sqrt{2}$

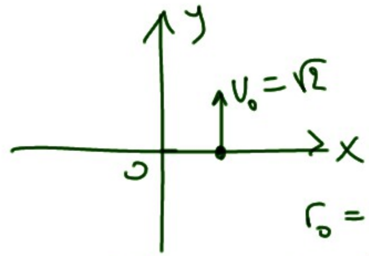
$u'' + u = -12u^5 + 10u^3$

$\ddot{x} = f(x)$

$u'u'' = (-12u^5 + 10u^3 - u)u'$

$\ddot{x} = \dfrac{d\dot{x}}{dx}\dot{x}$

$\dfrac{u'^2}{2} = -2u^6 + \dfrac{5}{2}u^4 - \dfrac{u^2}{2} + C$



Αρχικά $u\big|_{\varphi=0} = 1$, $u'\big|_{\varphi=0} = 0$

$u' = \dfrac{d(1/r)}{d\varphi} = \dfrac{-\frac{1}{r^2}\frac{dr}{dt}}{L/mr^2} = 0$

άρα $0 = -2 + \dfrac{5}{2} - \dfrac{1}{2} + C \Leftrightarrow C = 0$

$u'^2 = -4u^6 + 5u^4 - u^2$

$b = 5 \quad c = -4$

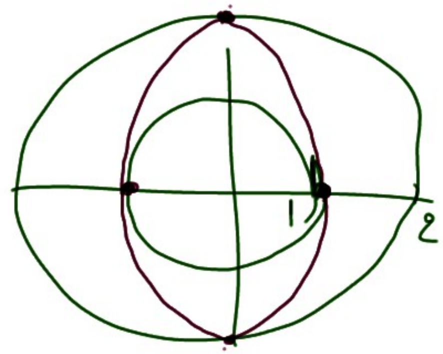
$\dfrac{du}{d\varphi} = \pm\sqrt{-4u^6 + 5u^4 - u^2} \Leftrightarrow \displaystyle\int \frac{du}{\sqrt{-4u^6 + 5u^4 - u^2}} = \pm\int d\varphi \Leftrightarrow$

$\Leftrightarrow \dfrac{1}{2}\arcsin\dfrac{5 - 2/u^2}{3} = \pm(\varphi + D) \Leftrightarrow \dfrac{5 - 2r^2}{3} = \pm\sin(2\varphi + 2D) = \sin(2\varphi + D_0)$, $D_0 = \dfrac{\pi}{2}$

(Αλλιώς : $\frac{\dot{r}^2}{2} + V_{eff}(r) = E = 0$ $\qquad$ $\dot{r} = \frac{dr}{d\varphi}\dot{\varphi} = \frac{dr}{d\varphi}\frac{L}{mr^2} = \frac{dr}{d\varphi}\frac{\sqrt{2}}{r^2}$)

$$r = \sqrt{\frac{5-3\cos(2\varphi)}{2}} = \sqrt{1+3\sin^2\varphi}$$

$$\cos(2\varphi) = \cos^2\varphi - \sin^2\varphi = 1 - 2\sin^2\varphi$$



$x = r\cos\varphi$

$y = r\sin\varphi$

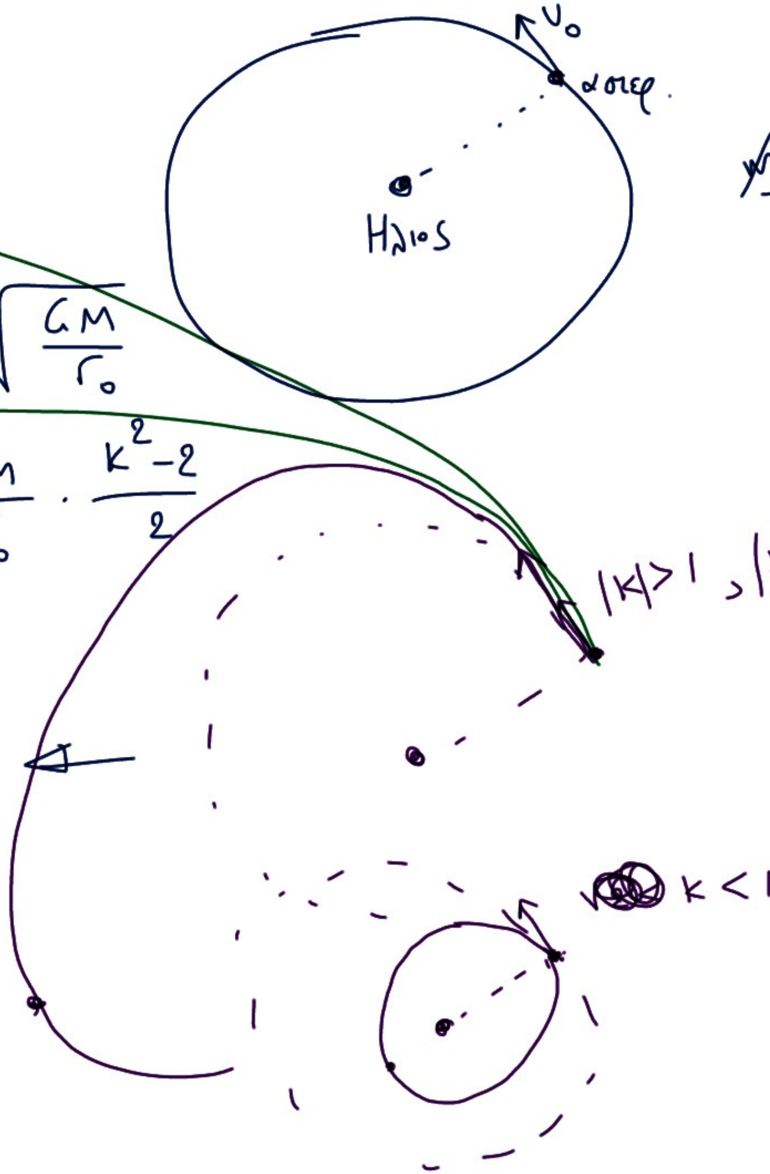
περίοδος ως προς φ είναι 2π.

Η τροχιά περιοδική

$$T = \int \frac{dt}{dr}dr = 4\int_1^2 \frac{dr}{\dot{r}} \quad ή \quad T = \int_0^{2\pi}\frac{d\varphi}{\dot{\varphi}} \quad με \quad \dot{\varphi} = \frac{L}{mr^2} = \frac{2\sqrt{2}}{5-3\cos(2\varphi)}$$

$$T = \int_0^{2\pi}\frac{5-3\cos(2\varphi)}{2\sqrt{2}}\,d\varphi = \frac{5}{\sqrt{2}}\pi .$$

Θέμα 3 εξέτασης 16/5/2016



$$\frac{\cancel{m}v^2}{r} = \frac{GM\cancel{m}}{r^2}$$

Πριν την κρούση $v_0 = \sqrt{\frac{GM}{r_0}}$

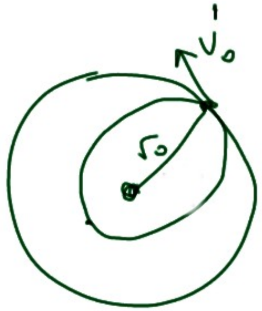
Μετά την κρούση $v_0' = k v_0 = k\sqrt{\frac{GM}{r_0}}$

$$E = \frac{1}{2} m v_0'^2 - \frac{GMm}{r_0} = \frac{GMm}{r_0} \cdot \frac{k^2-2}{2}$$

$$L = m r_0 v_0' = k m \sqrt{G M r_0}$$

- έλλειψη $E<0 \Leftrightarrow |k|<\sqrt{2}$
- παραβολή $E=0 \Leftrightarrow |k|=\sqrt{2}$
- υπερβολή $E>0 \Leftrightarrow |k|>\sqrt{2}$

(γ) $k^2 = \frac{1}{2}$, $v_0' = \sqrt{\frac{GM}{2r_0}}$



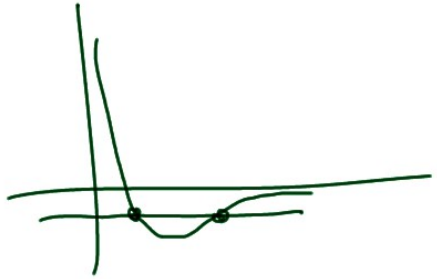
$$m\Omega^2 r = \frac{GMm}{r^2}$$

$$\Omega = \sqrt{\frac{GM}{\alpha^3}} \quad , \quad T = \frac{2\pi}{\Omega}$$

$$\Delta t = \frac{1}{2} T$$

$$T = \frac{2\pi \alpha^{3/2}}{\sqrt{GM}} \quad (1)$$

$$\alpha = \frac{\overset{1\,AU}{r_\pi} + \overset{3\,AU}{r_\alpha}}{2} = 2\ AU$$



$$1 \text{ έτος} = \frac{2\pi}{\sqrt{GM}} (1\,AU)^{3/2} \quad (2)$$

$$\frac{(1)}{(2)} \rightarrow \frac{T}{1 \text{ έτος}} = \left(\frac{\alpha}{1\,AU}\right)^{3/2} \text{ δηλ } T = \alpha^{3/2} = 2^{3/2}$$

$$\Delta t = \sqrt{2} \text{ έτη} = 1.4 \text{ έτη}$$