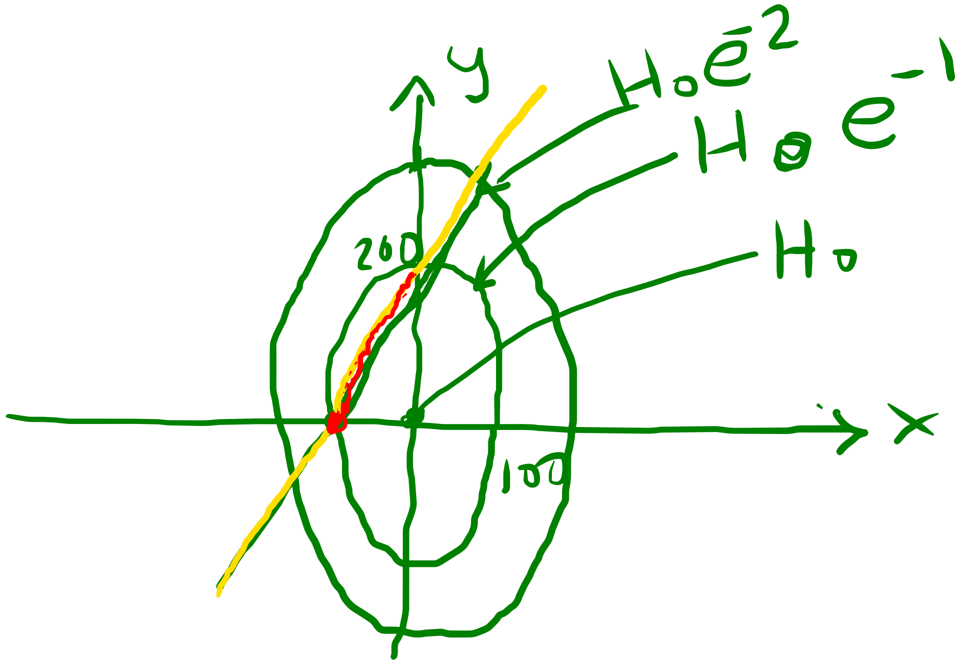
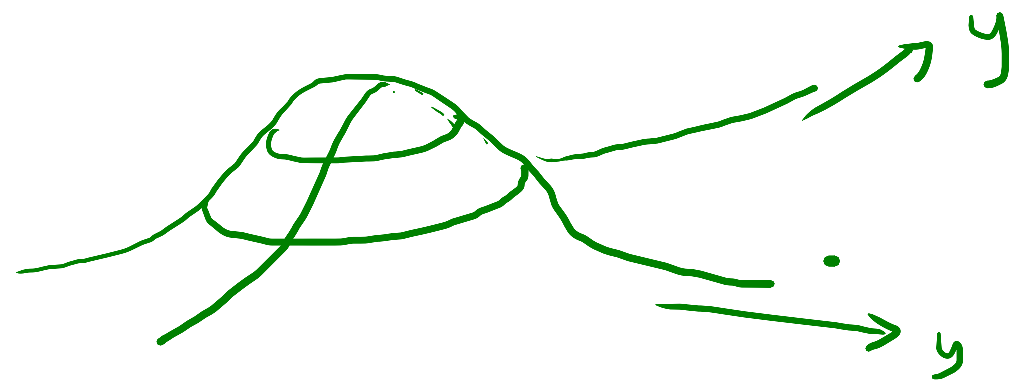
Πέμπτη 26-5 (18)

- πολλαπλασιαστές Lagrange
- συστήματα συστήματα με δεσμούς
- L με πολ/στές Lagrange
- βαθμοί ελευθερίας
- προβλήματα

$$H(x,y) = H_0 e^{-\left[\left(\frac{x}{100}\right)^2 + \left(\frac{y}{200}\right)^2\right]}$$





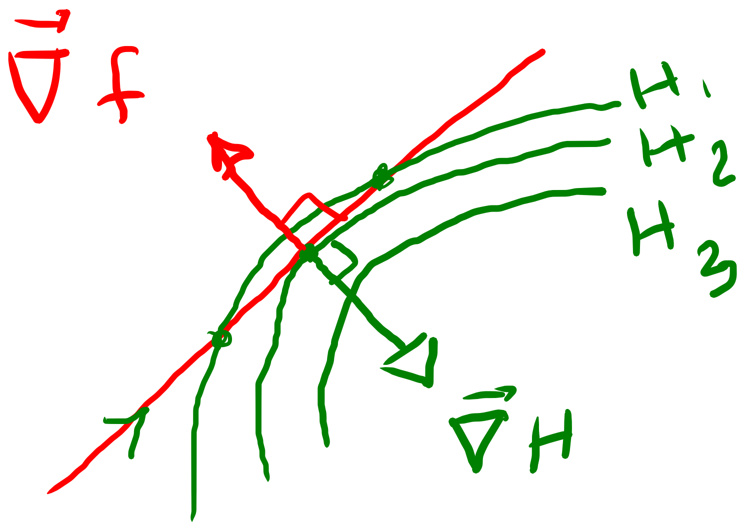
$$y = 2(x+100)$$

ερώτημα ποιό είναι το μεγαλύτερο υψόμετρο που θα φτάσουμε?

$$H(x,y) \xrightarrow[\text{με δεσμό}]{\max}$$

$$y-2x-200=0$$

$f = y-2x-200$ ο δεσμός



$H_3 > H_2 > H_1$

$\vec{\nabla} H =$ διάνυσμα $\perp$ στις $H =$ σταθ. με φορά προς τα μεγαλ. H.

$$\vec{\nabla}(y-2x-200) = (-2, 1)$$

$$\underline{\vec{\nabla} f} = \lambda \, \underline{\vec{\nabla} H}$$

$$\vec{\nabla} H = H_0 e^{-[\quad]} \left(-2\frac{x}{100^2} , -2\frac{y}{(200)^2} \right)$$

$$\vec{\nabla} f = (-2, 1)$$

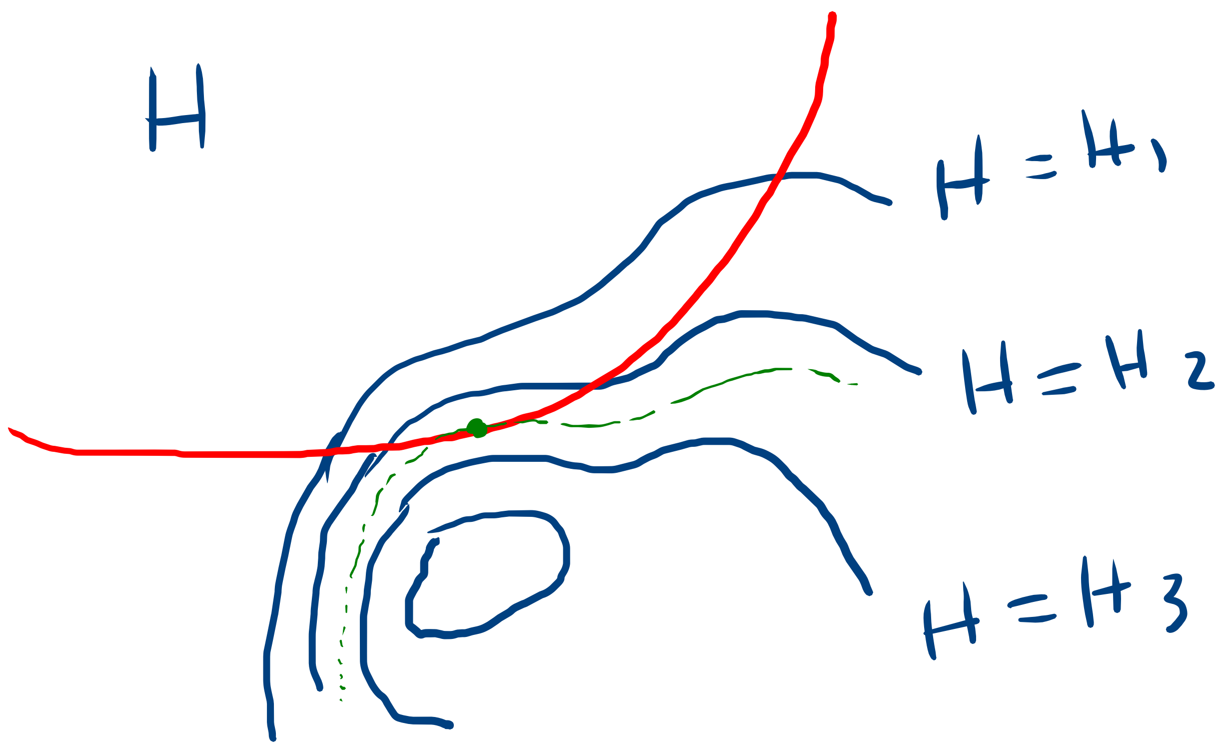
$$\vec{\nabla} H = \lambda \vec{\nabla} f \implies \left(\frac{-x}{(100)^2}, \frac{-y}{(200)^2} \right) \parallel (-2, 1)$$

$$\frac{\cancel{-} x/(1\cancel{00})^2}{\cancel{-} y/(2\cancel{00})^2} = \frac{-2}{1} = \frac{4x}{y} \implies y_0 = -2x_0$$

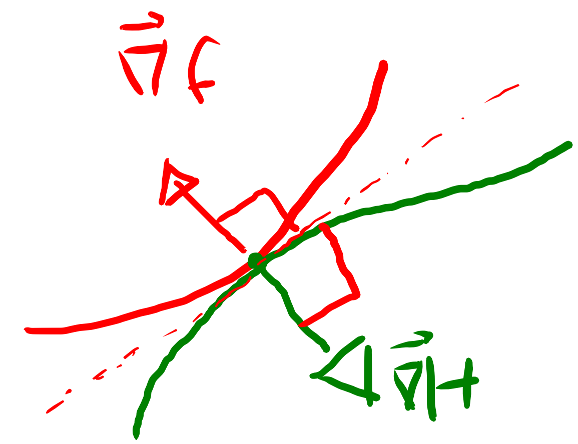
$$y_0 - 2x_0 - 200 = 0$$

$$-4x_0 - 200 = 0 \implies \boxed{\begin{matrix} x_0 = -50 \\ y_0 = 100 \end{matrix}}$$

H



f

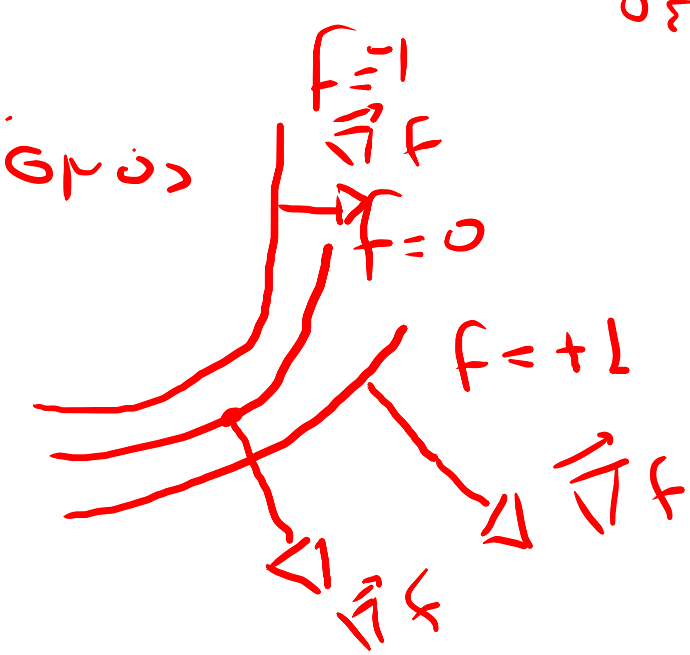


δεν έχει σημείο αν $\lambda > 0$ ή $\lambda < 0$

$f = 0$ είναι ο δεσμός

$f = 1$

$f = -1$



$$H(x, y = 2x + 200) = \ldots\ldots$$

$$\frac{df}{dx} = 0 = \ldots\ldots$$ λύση

$$\log(yx) + \sin(x+y) = e^{x^2} \Rightarrow y = y(x)$$

ακατόρθ. ~~δύσκολο~~
αναλυτικά

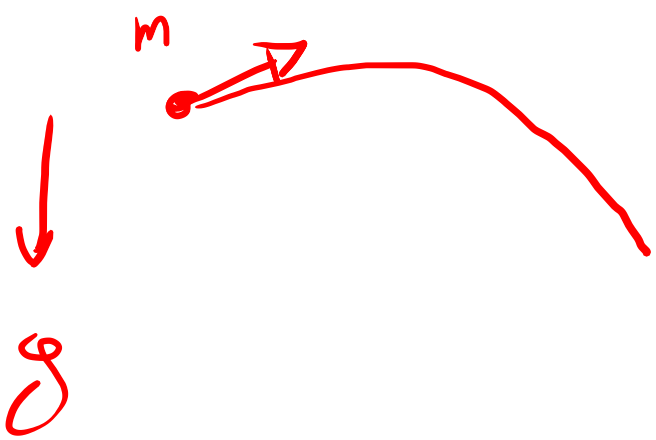
$$f = \log(xy) + \sin(x+y) - e^{x^2}$$

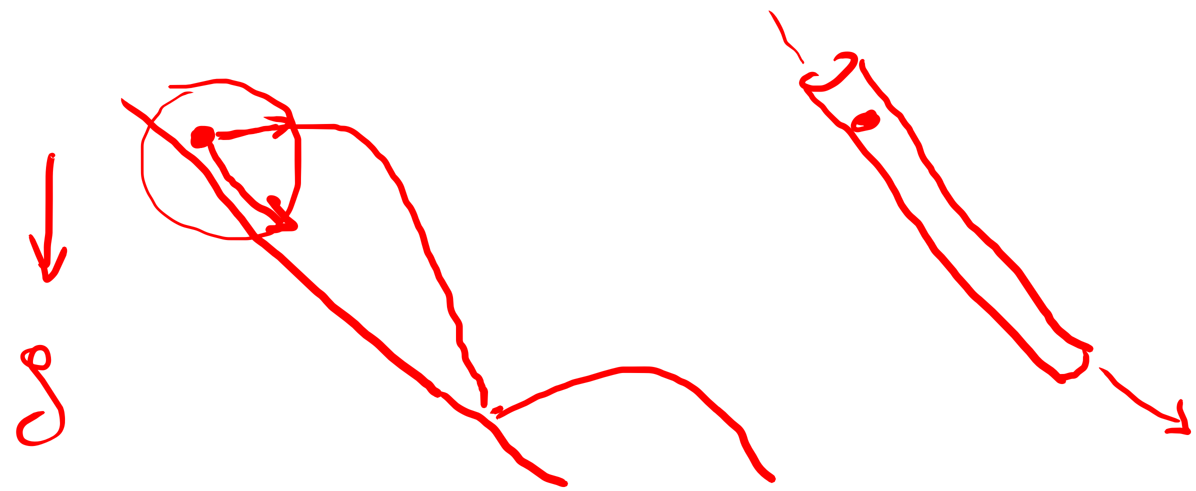
$$\vec{\nabla} f = \lambda \vec{\nabla} H \ldots\ldots$$ $x_0 \leftrightarrow y_0$

Βρες λ : $\vec{\nabla}(H - \lambda f) = 0$

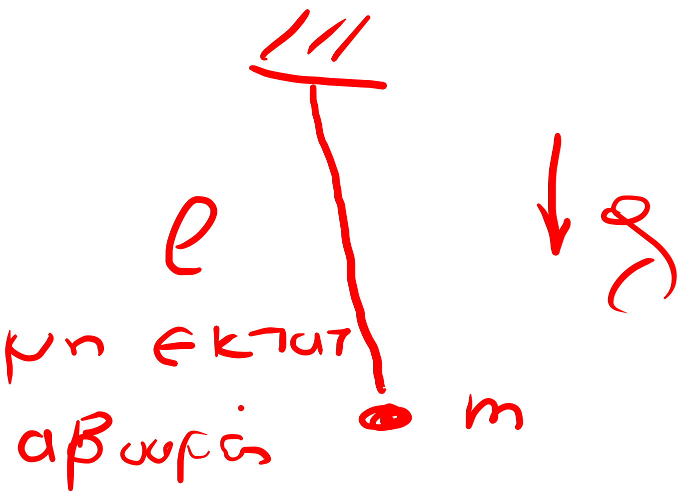
Φυσικά προβλ. με δεσμούς

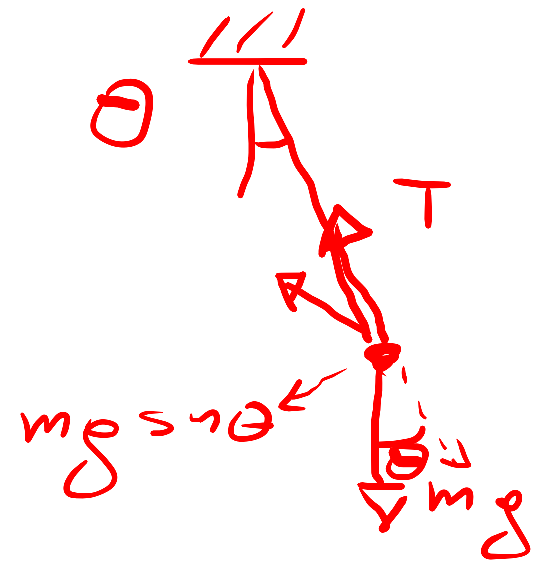
L : αν γνωρίζω το δυναμικό που περιγράφει τις δυνάμεις που υπεισέρχ. στο πρόβλ.



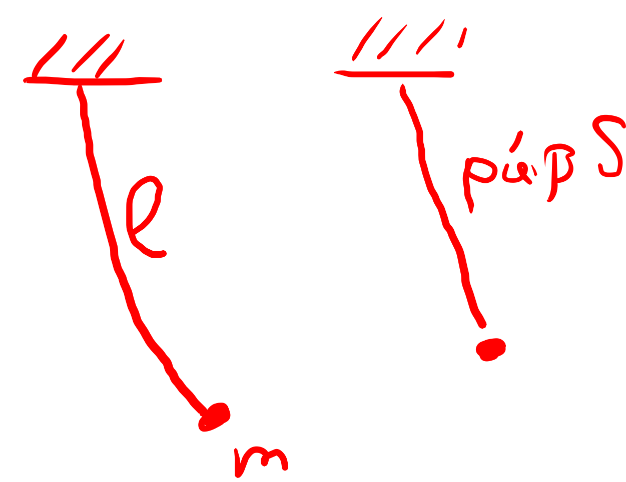


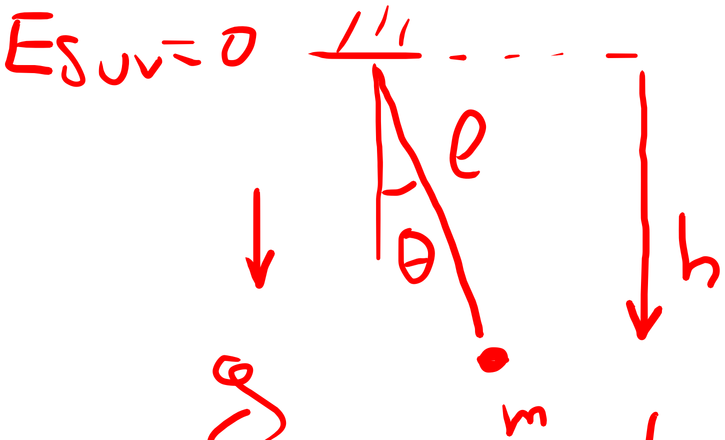
η νευτώνεια εικόνα ενός εκκρεμούς





$$-mg\ell \sin\theta = m\ell^2\ddot{\theta}$$

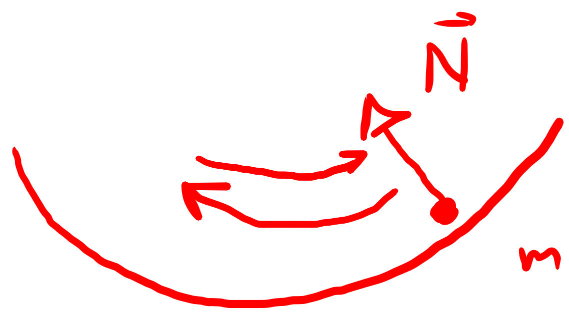




$h = l\cos\theta$

$$L = \frac{1}{2} m (l\dot{\theta})^2 - (-mgl\cos\theta)$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

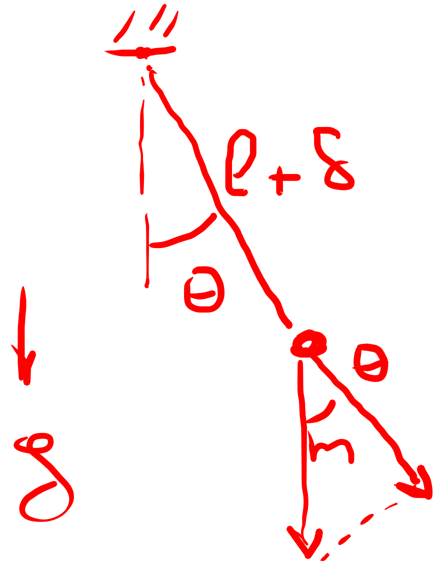


$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

EL: $ml^2\ddot{\theta} = -mgl\sin\theta$

$$\ddot{\theta} = -g/l \, \sin\theta$$



$B_{\parallel} = mg\cos\theta$

$\delta \to 0$

$V(\delta)$ άγνωστη

θα την βάλω στη Lagr.

$$L = \frac{1}{2} m \left[(\ell+\delta)^2 \dot{\theta}^2 + \dot{\delta}^2 \right] + mg(\ell+\delta)\cos\theta - V(\delta)$$

$$\theta: \left[m(\ell+\delta)^2 \dot{\theta} \right]^{\cdot} = -mg(\ell+\delta)\sin\theta \xrightarrow{\delta \to 0} m\ell^2 \ddot{\theta} = -mg\ell\sin\theta$$

$$\delta: \left[m\dot{\delta} \right]^{\cdot} = m(\ell+\delta)\dot{\theta}^2 + mg\cos\theta - V'(\delta)$$

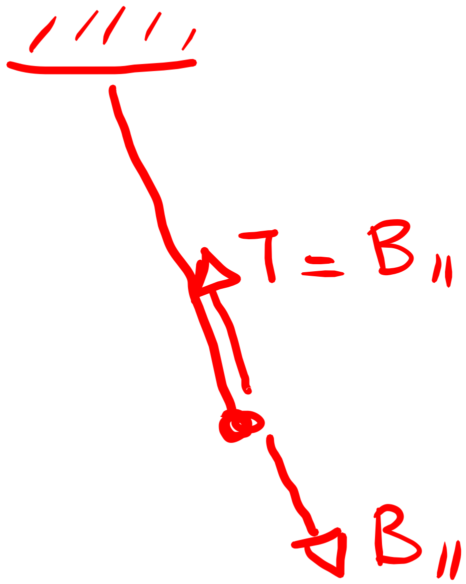
$$\xrightarrow{\delta \to 0} \quad 0 = m\ell\dot{\theta}^2 + mg\cos\theta - V'(\delta=0)$$

T η τάση της ράβδου

$B_{\parallel}$ στο νήμα

$$T = -m\ell\dot{\theta}^2 - mg\cos\theta$$

$$mv^2/\ell = m(\ell\dot{\theta})^2/\ell$$



$T = B_{\parallel} + m\upsilon^2/\ell$

$(\vec{T} + \vec{B})_{\parallel} = \dfrac{m\upsilon^2}{\ell}$ = κεντρομ.

πολ/στής Lagr.

$$L = \frac{1}{2}m\left(\dot r^2 + r^2\dot\theta^2\right) + mgr\cos\theta + \underbrace{\lambda(r-\ell)}_{\text{εξίσωση δεσμού}}$$

$-V(\delta)$

$$L(r, \theta, \dot r, \dot\theta, \lambda, \cancel{\dot\lambda})$$

EL: λ: $\dfrac{\partial L}{\partial \lambda} = 0 \Rightarrow r - \ell = 0 \Rightarrow \underline{r = \ell}$ (1)

$r = \ell$

$r - \ell = 0$

r: $m\ddot r = \dfrac{\partial L}{\partial r} = mr\dot\theta^2 + mg\cos\theta + \lambda$ (2)

θ: $(mr^2\dot\theta)^{\cdot} = \dfrac{\partial L}{\partial \theta} = -mgr\sin\theta$ (3) $\Rightarrow m\ell^2\ddot\theta = -mg\ell\sin\theta$ ✓

(2) $0 = m\ell\dot\theta^2 + mg\cos\theta + \lambda \Rightarrow \underline{\lambda = -m\ell\dot\theta^2 - mg\cos\theta}$

$\Downarrow$

δύναμη λόγω ικανοπ. του δεσμού

$$\boxed{\vec{F}_{\delta\varepsilon\sigma\mu} = \lambda \vec{\nabla}(\delta\varepsilon\sigma\mu o\acute{\upsilon})} = \lambda \hat{r}$$

$r - \ell = 0$

$$= \left(- mg\cos\theta - m\ell\dot{\theta}^2\right)\hat{r}$$

$$= (-\hat{r})\left(B_{\parallel} + \kappa\varepsilon\nu\tau\rho o\mu\right)$$

$$L \xrightarrow[Lag]{\pi o\lambda} L + \underbrace{\lambda(\delta\varepsilon\sigma\mu o\acute{\upsilon})}_{-V(\delta\varepsilon\sigma\mu o\acute{\upsilon})}$$

$$\vec{F}_{\delta\varepsilon\sigma\mu} = -\vec{\nabla} V(\delta\varepsilon\sigma\mu) = \lambda \vec{\nabla}\left(\begin{matrix}\varepsilon\xi\acute{\iota}\sigma.\\ \delta\varepsilon\sigma\mu o\acute{\upsilon}\end{matrix}\right)$$

Προβλ. $\vec{B} = B\hat{z}$ $\quad B=$ σταθ. $\quad$ δεσμ. ένα σωματίδιο φορτ.

$z=0\,,\quad x^2+y^2=R^2$

$$L = \frac{1}{2}m\left(\dot{\underline{r}}^2 + r^2\dot{\underline{\theta}}^2 + \dot{\underline{z}}^2\right) + q\vec{A}\cdot\vec{v} + \underline{\lambda}(z-0) + \underline{\mu}(r-R)$$

$$\boxed{\vec{A} = \frac{B}{2}(-y\,,x\,,0)}\ ?$$

$$\vec{\nabla}\times\vec{A} = \frac{B}{2}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = \left(0\hat{x} + 0\hat{y} + 2\hat{z}\right)\frac{B}{2}$$

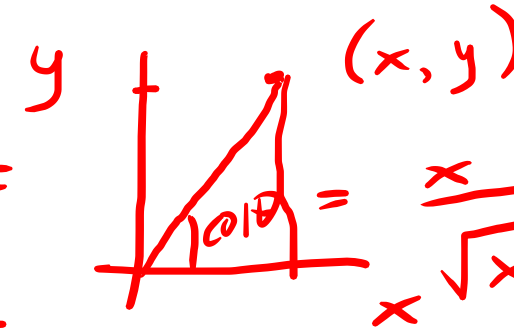
$$= B\hat{z}$$

$$\vec{A} = \frac{B}{2}(-y, x, 0) = \frac{B\overbrace{\sqrt{x^2+y^2}}^{r}}{2}\underbrace{\left(\underbrace{\frac{-y}{\sqrt{x^2+y^2}}}_{\sin\theta}, \underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\cos\theta}, 0\right)}_{\hat{\theta}}$$

$$\hat{r} = \hat{x}\cos\theta + \hat{y}\sin\theta$$

$$\hat{\theta} = \perp\hat{r} = \underline{-\hat{x}\sin\theta + \hat{y}\cos\theta}$$

$\sin\theta =$



$= \frac{x}{\sqrt{x^2+y^2}}$

$\frac{y}{\sqrt{x^2+y^2}}$

$$\vec{A} = \frac{Br}{2}\hat{\theta} \quad ✓$$

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right) + q \frac{Br}{2} \underbrace{\hat{\theta} \cdot \vec{v}}_{r\dot{\theta}}$$

$$+ \lambda z + \mu (r - R)$$

$$\vec{v} = \dot{r} \hat{r} + r\dot{\theta} \hat{\theta} + \dot{z} \hat{z}$$

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right) + \frac{qBr^2}{2} \dot{\theta} + \lambda z + \mu (r - R)$$

(1) : λ : $\frac{\partial L}{\partial \lambda} = 0 \Rightarrow z = 0$

(2) : μ $\Rightarrow r = R$

(3) r : $m\ddot{r} = \frac{\partial L}{\partial r} = m r \dot{\theta}^2 + qBr\dot{\theta} + \mu$

(4) θ : $\left(m r^2 \dot{\theta} \right)^{\cdot} = 0 \Rightarrow m r^2 \dot{\theta} = \text{const}$

(5) z : $m\ddot{z} = \frac{\partial L}{\partial z} = \lambda$

$$z=0 \, , \quad r=R$$

$$m\ddot{r} = mr\dot{\theta}^2 + qBr\dot{\theta} + \mu \;\Rightarrow\; 0 = mR\dot{\theta}^2 + qBR\dot{\theta} + \mu$$

$$mr\dot{\theta}^2 = \text{ثابت} \;\longrightarrow\; mR\dot{\theta}^2 = \text{ثابت} \;\Rightarrow\; \dot{\theta} = \text{ثابت}$$

$$m\ddot{z} = \lambda \;\Rightarrow\; \boxed{\lambda = 0}$$

$$\mu = -mR\dot{\theta}^2 - qBR\dot{\theta}$$

$$= R\dot{\theta}\left(-m\dot{\theta} - qB\right)$$

$$= -mR\dot{\theta}\left(\dot{\theta} + \frac{qB}{m}\right)$$

$r = R$ ✓

$z = 0$ ✓

$\lambda = 0$

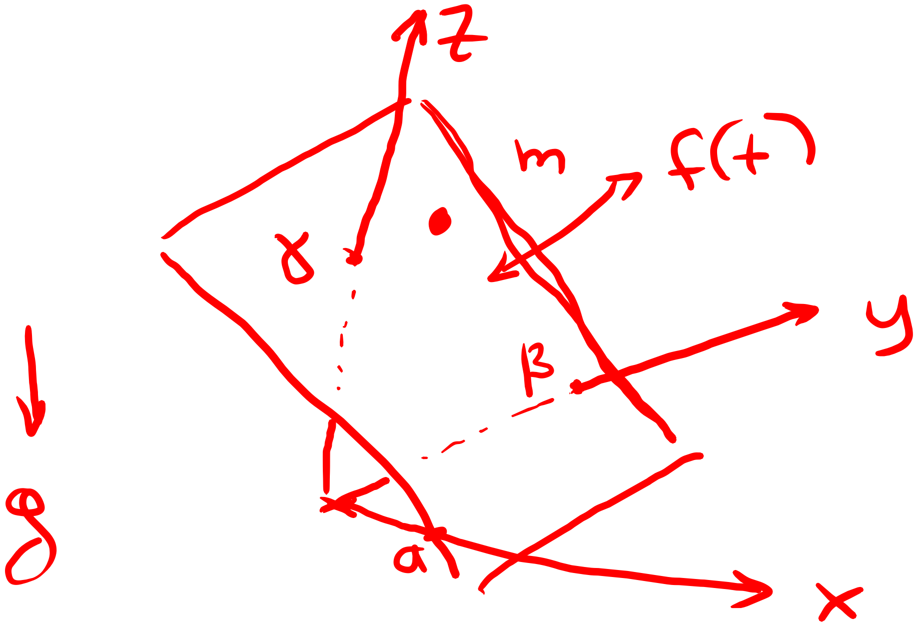
$\dot{\theta} = \text{ثابت}$ ✓

$$\mu = -mR\dot{\theta}\left(\dot{\theta} + \frac{qB}{m}\right)$$

$$\vec{F}_{\text{قید}} = \lambda \vec{\nabla}(z) + \mu \vec{\nabla}(r-R)$$

$$= \lambda \hat{z} + \mu \hat{r}$$

$$= -mR\dot{\theta}\left(\dot{\theta} + \frac{qB}{m}\right)\hat{r}$$



$$\frac{x}{a} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$$

$$= f(t)$$

$$L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mg\underline{z} + \lambda\left[\frac{x}{a} + \frac{y}{\beta} + \frac{z}{\gamma} - f(t)\right]$$

(1) λ: $\left[\frac{x}{a} + \frac{y}{\beta} + \frac{z}{\gamma} - f(t)\right] = 0$ ✓

(2) x: $m\ddot{x} = \frac{\partial L}{\partial x} = \lambda/a \longrightarrow x = \frac{\lambda}{am} t^2 + At + B$ ואם

$\lambda = \lambda(t)$ סד ידוע γ

(3) y: $m\ddot{y} = \lambda/\beta$

(4) z: $m\ddot{z} = -mg + \lambda/\gamma$

$$\underbrace{m\left(\frac{\ddot{x}}{a}+\frac{\ddot{y}}{\beta}+\frac{\ddot{z}}{\gamma}\right)}_{\ddot{f}(t)} = \frac{\lambda}{a^2}+\frac{\lambda}{\beta^2}+\frac{\lambda}{\gamma^2}-\frac{mg}{\gamma}$$

$$\frac{x}{a}+\frac{y}{\beta}+\frac{z}{\gamma}=f(t)$$

αν $f(t)$ είναι γραμ.

τότε το $\ddot{f}=0$

$$\lambda\left(\frac{1}{a^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)-\frac{mg}{\gamma}=m\ddot{f}$$

$$\lambda(t)=\frac{m(\ddot{f}+g/\gamma)}{\left(\frac{1}{a^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)}$$

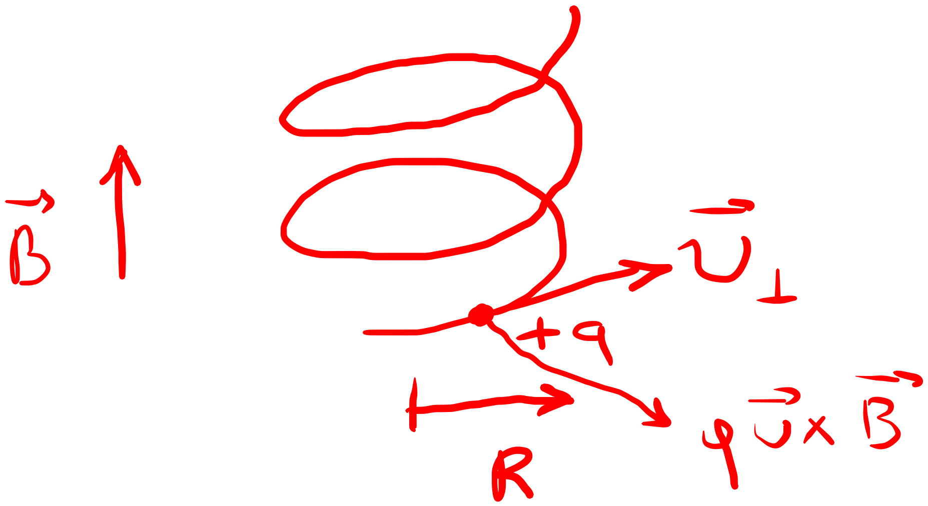
$(\quad)$

$$\ddot{x}=\frac{\lambda(t)}{am} \rightarrow x(t)=\iint\frac{\lambda(t)\,dt}{am}dt$$

αν $\ddot{f}=0$ αν κιν. με ομαλ.

κιν. το επ.π.

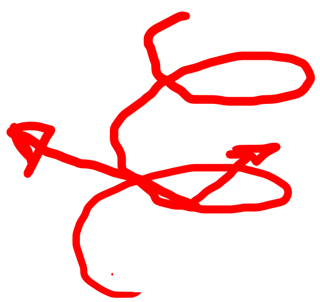
$$\ddot{x}=\frac{\cancel{m}g/\gamma}{(\quad)a\cancel{m}} \Rightarrow x(t)=\frac{1}{2}\frac{\cancel{m}g/\gamma\, t^2}{(\quad)a\cancel{m}}+At+B$$



$$qBv_\perp = m v_\perp^2 / R$$

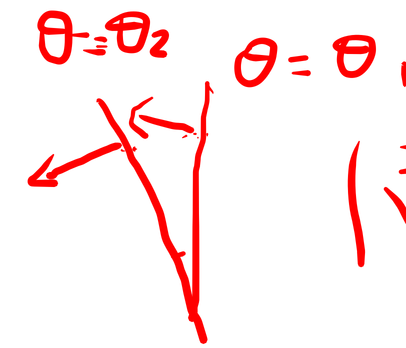
$$\cancel{R} = \frac{m v_\perp}{qB} = \frac{m \cancel{R} \dot{\theta}}{qB}$$

$$\dot{\theta} = \frac{(-)qB}{m}$$



$\dot{\theta} < 0$

$$\vec{\nabla} r = \hat{r} \qquad \underline{\vec{\nabla} \theta} \neq \hat{\theta} \qquad \vec{\nabla} \theta = \frac{1}{r} \hat{\theta}$$



$$|\vec{\nabla} f| \propto (\text{αποσταση ισουψων})^{-1}$$