

Quasi-linear PDEs (III)

Conservation laws, shock dynamics
and weak solutions

Conservation laws

A conservation law has the general form:

$$\frac{\partial}{\partial t} T(x, t) + \frac{\partial}{\partial x} X(x, t) = 0$$

where $T(x, t)$ is the **density of the conserved quantity** and $X(x, t)$ is the **associated flux**.

Integrating with respect to x , the conservation law can be expressed in the following **integral form**:

$$\frac{\partial}{\partial t} \int_{x_1}^{x_2} T(x, t) dx = -X(x, t) \Big|_{x=x_1}^{x_2}$$

i.e., the rate of change of the density over an interval depends only on the flux through its end points

If $X(x_2, t) = X(x_1, t)$ then $\int_a^b T(x, t) dx = \text{const.}$ **conserved quantity**

Examples

$$1) \text{ Linearized KdV equation: } u_t + u_{xxx} = 0 \quad (1)$$

This equation is already in the form of conservation law:

$$(u)_t + (u_{xx})_x = 0, \quad T_1 = u, \quad X_1 = u_{xx}$$

$$\int_{-\infty}^{+\infty} T_1 dx = \int_{-\infty}^{+\infty} u dx = \text{mass} = \text{const.};$$

Also, from **Parseval's theorem**, linear dispersive equations with real $\omega(k)$ **conserve the energy** **[Can you show this?]**

Indeed, multiplying (1) by u we find:

$$(u^2)_t + \left(uu_{xx} - \frac{1}{2} u_x^2 \right)_x = 0, \quad T_2 = u^2, \quad X_2 = uu_{xx} - \frac{1}{2} u_x^2$$

$$\int_{-\infty}^{+\infty} T_2 dx = \int_{-\infty}^{+\infty} u^2 dx = \text{energy} = \text{const.}$$

Examples – cont.

2) Schrödinger equation: $i\psi_t + \frac{1}{2}\psi_{xx} - V(x)\psi = 0 \quad (2)$

Multiplying (2) by $\bar{\psi}$, its c.c. by ψ , and adding, we find:

$$(|\psi|^2)_t + \left(\frac{i}{2} (\bar{\psi}\psi_x - \psi\bar{\psi}_x) \right)_x = 0, \quad T = |\psi|^2,$$

$$\int_{-\infty}^{+\infty} |\psi|^2 dx = \text{const.},$$

$$X = \frac{i}{2} (\bar{\psi}\psi_x - \psi\bar{\psi}_x) = \text{current}$$

3) Hopf equation: $u_t + uu_x = 0, \quad (3)$

$$(u)_t + \left(\frac{1}{2} u^2 \right)_x = 0, \quad T = u, \quad X = \frac{1}{2} u^2,$$

Additional comments on the Hopf equation

Consider the Hopf equation: $u_t + uu_x = 0$, $u(x,0) = f(x)$
and assume that $u(x,t)$ represents a density; then:

$$M_{x_1, x_2} = \int_{x_1}^{x_2} u(x,t) dx : \text{total mass in } x \in [x_1, x_2]$$

Using the conservation law and its integral form we obtain:

$$(u)_t + \left(\frac{1}{2} u^2 \right)_x = 0 \Rightarrow \frac{d}{dt} \int_{x_1}^{x_2} u dx = \underbrace{\frac{dM_{x_1, x_2}}{dt}}_{\text{rate of change of mass}} = \underbrace{\frac{1}{2} u^2(x_1, t) - \frac{1}{2} u^2(x_2, t)}_{\text{net mass / flux through the endpoints}}$$

E.g., in the **traffic flow** problem: $u(x,t) = \text{traffic density}$;

**Rate of change of # of vehicles in $[x_1, x_2]$ =
{# of vehicles entering x_1 } - {# of vehicles leaving at x_2 }**

Additional comments on the Hopf equation

Assume that the initial data, $f(x)$, has a **finite mass**:

$$u(x,0) = f(x), \quad \left| \int_{-\infty}^{+\infty} f(x,t) dx \right| < \infty$$

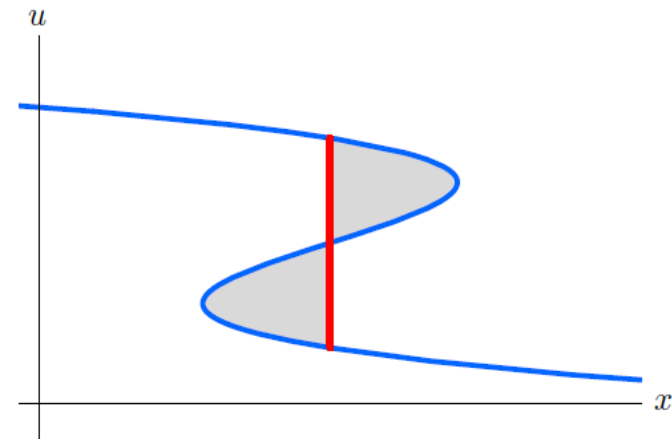
i.e., $f(x) \rightarrow 0$ rapidly as $x \rightarrow \infty$. Then,

$$\int_{-\infty}^{+\infty} u(x,t) dx = \int_{-\infty}^{+\infty} u(x,0) dx = \int_{-\infty}^{+\infty} f(x) dx = \text{const.}$$

i.e., mass remains constant - and is equal to its initial value

What happens in the case of a shock, where $u(x,t)$ displays a **discontinuity**?

Equal Area Rule: it ensures that the total mass of the shock wave solution remains constant



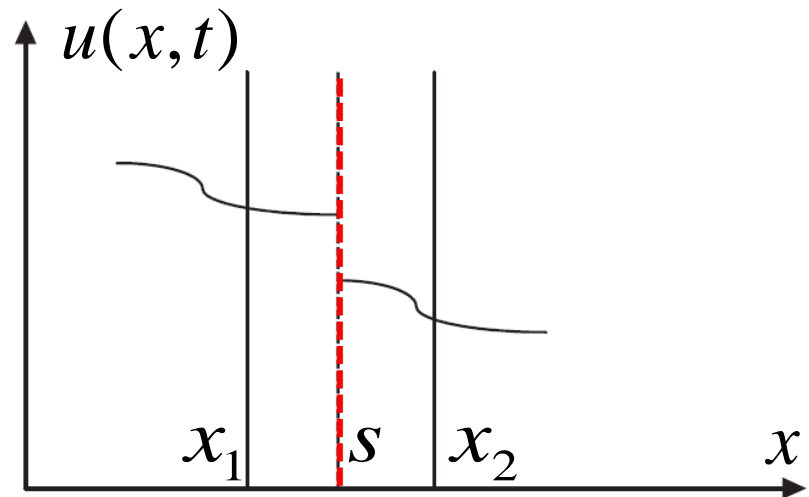
Shock waves and conservation laws

- We wish to take the solution further in time, **beyond** $t = t_B$, but **do not want multi-valued solutions** for **physical** and **mathematical** reasons
- Discontinuities, i.e., shocks, can be introduced by considering the Hopf equation as coming from the above mentioned **conservation law** and its corresponding **integral form**:

$$u_t + \left(\frac{1}{2} u^2 \right)_x = 0 \Rightarrow \frac{d}{dt} \int_{x_1}^{x_2} u(x, t) dx + \frac{1}{2} [u^2(x_2, t) - u^2(x_1, t)] = 0$$

This equation can support a shock wave since it is an **integral relation**.

Assume that between two points x_1 and x_2 we have a **discontinuity** that can change in time, $x = s(t)$



Rankine-Hugoniot (RH) condition (I)

Then, we rewrite the integral form of the conservation law as:

$$\frac{d}{dt} \left[\int_{x_1}^{s(t)} u(x, t) dx + \int_{s(t)}^{x_2} u(x, t) dx \right] + \frac{1}{2} \left[u^2(x_2, t) - u^2(x_1, t) \right] = 0$$

Let: $\begin{cases} \underline{x_1 = s(t) - \varepsilon} \\ \underline{x_2 = s(t) + \varepsilon} \end{cases}$ with $\varepsilon > 0$.

Next, using **Leibnitz integral rule** we obtain:

$$\begin{aligned} & \cancel{u(s, t) \frac{ds}{dt}} - u(s - \varepsilon, t) \frac{ds}{dt} + \int_{s - \varepsilon}^s u_t(x, t) dt \\ & + u(s + \varepsilon, t) \frac{ds}{dt} - \cancel{u(s, t) \frac{ds}{dt}} + \int_s^{s + \varepsilon} u_t(x, t) dt \\ & + \frac{1}{2} \left[u^2(s + \varepsilon, t) - u^2(s - \varepsilon, t) \right] = 0 \end{aligned}$$

Rankine-Hugoniot (RH) condition (II)

Thus:
$$\frac{ds}{dt} [u(s + \varepsilon, t) - u(s - \varepsilon, t)] + \int_{s-\varepsilon}^s u_t dt + \int_s^{s+\varepsilon} u_t dt$$

$$+ \frac{1}{2} [u^2(s + \varepsilon, t) - u^2(s - \varepsilon, t)] = 0$$

In the limit $\varepsilon \rightarrow 0$:
$$\left\{ \begin{array}{l} \underline{x_1 = s - \varepsilon \rightarrow x^-}, \quad \underline{x_2 = s + \varepsilon \rightarrow x^+} \\ \int_{s-\varepsilon}^s u_t dt + \int_s^{s+\varepsilon} u_t dt = \int_{s-\varepsilon}^{s+\varepsilon} u_t dt \rightarrow 0 \end{array} \right.$$

Hence:

$$\frac{ds}{dt} [u(x^-, t) - u(x^+, t)] + \frac{1}{2} [u^2(x^+, t) - u^2(x^-, t)] = 0 \Rightarrow$$

$$\frac{ds}{dt} = \frac{1}{2} [u(x^+, t) + u(x^-, t)] = \frac{1}{2} (u^+ + u^-), \quad u^\pm = u(x^\pm, t)$$

speed of the shock in terms of the jump discontinuities

Generalization of the RH condition

Consider the Hopf equation: $u_t + c(u)u_x = 0$

and assume that F is the **antiderivative** of $c(u)$, i.e., $F'(u) = c(u)$

Then, the Hopf equation can be written as the **conservation law**:

$$u_t + \frac{\partial}{\partial x} (F(u)) = 0$$

In this case, the **Rankine-Hugoniot condition** take the form:

$$\frac{ds}{dt} = \frac{F(u_+, t) - F(u_-, t)}{u_+ - u_-}$$

or:

$$s'(t)[u] = [F(u)] \quad \text{Jump condition}$$

where $[g]$ denotes the **jump of g along a discontinuity**

Rankine-Hugoniot condition - history

■ They are named in recognition of work carried out by **William Rankine (Scottish)** in **1870** and **Pierre Henri Hugoniot (French)** in **1887**

■ They were originally found by **Sir George Stokes (Irish)** in **1848** and **Bernhard Riemann (German)** in **1860**

■ **Lord Kelvin (Irish)** and **Lord Rayleigh (English)** criticized Stokes' work.



W. Rankine



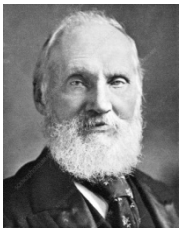
P. Hugoniot



G. Stokes



B. Riemann



Kelvin

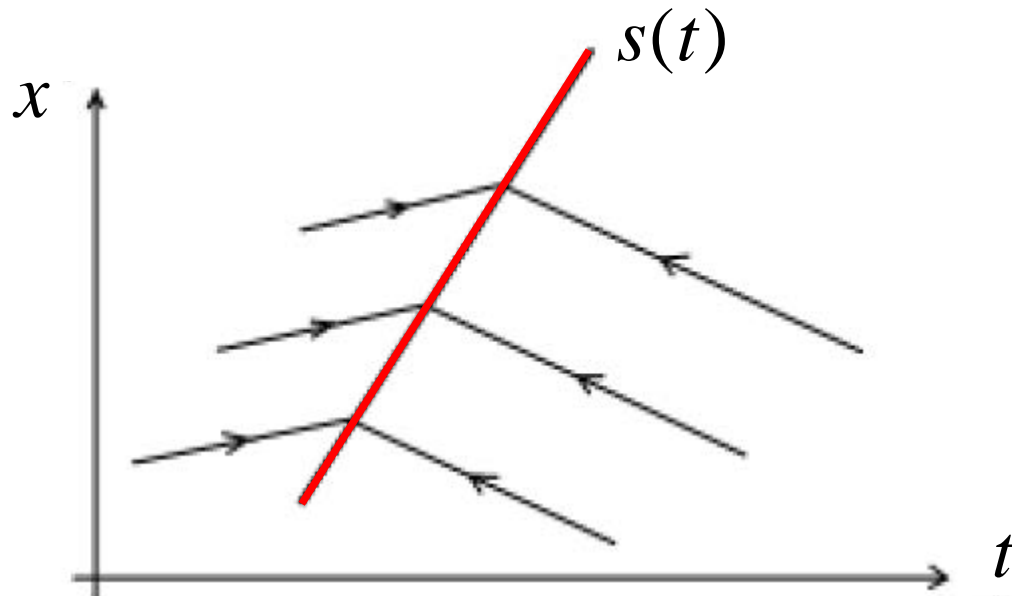


Rayleigh

■ Stokes thought that he was **wrong** and deleted that part when his collected works were published. **The missing part was restored in 1966**

RH condition and weak solutions (I)

- The **RH condition** (also referred to as **shock condition** or **jump condition**), are used to **avoid multi-valued behavior** in the solution, which would otherwise occur after characteristics cross
- We are interested in finding **weak** (or **generalized**) **solutions** featuring only **essential discontinuities**: the discontinuity curve $x = s(t)$ is essential if characteristics from each side of $x = s(t)$ intersect on $s(t)$ as t grows.



RH condition and weak solutions (II)

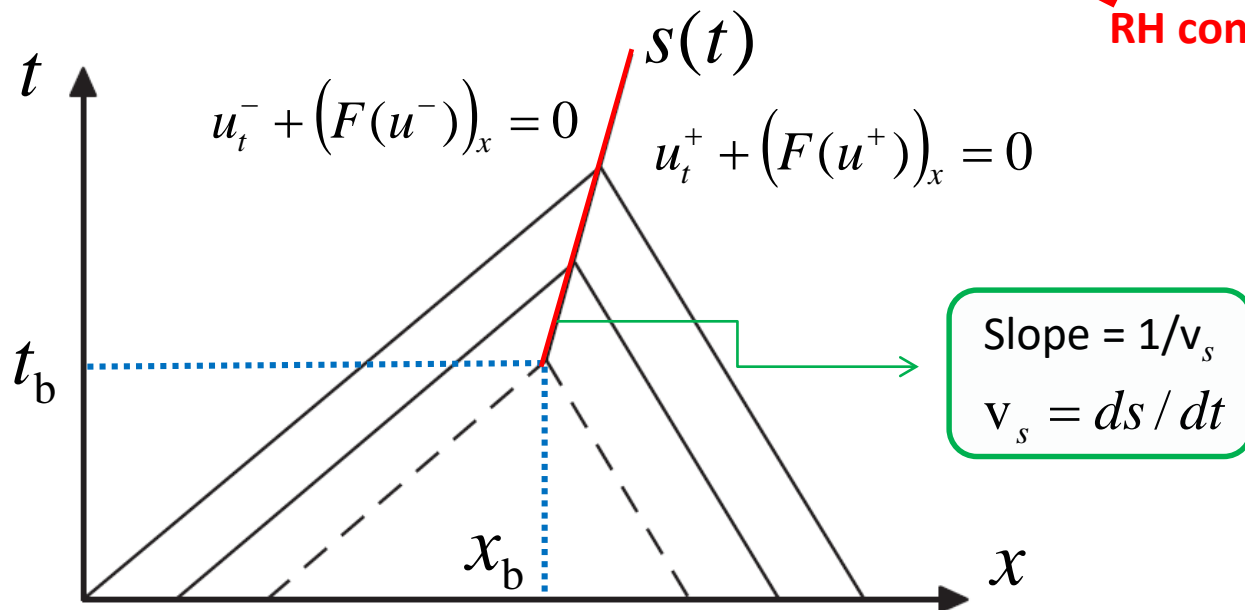
To be more specific, consider the Hopf equation: $u_t + (F(u))_x = 0$

Then, a **weak solution** of the Hopf equation, **valid for $t \geq t_b$** , is:

$$u = \begin{cases} u^-, & \text{for } (x, t) \text{ on the left of } s(t) \\ u^+ & \text{for } (x, t) \text{ on the right of } s(t) \end{cases}$$

with $s(t)$ satisfying the ODE: $\frac{ds}{dt} = \frac{F(u^+, t) - F(u^-, t)}{u^+ - u^-}$, $s(t_b) = x_b$

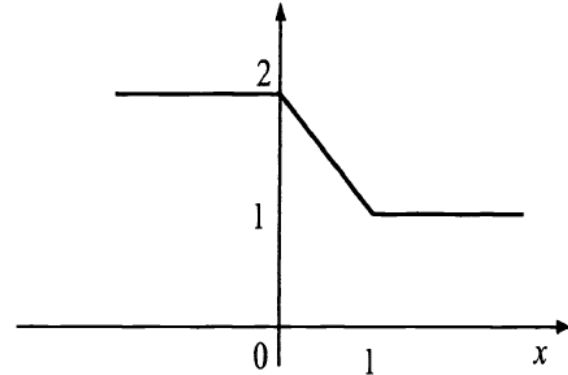
RH condition



Revisiting a previous example

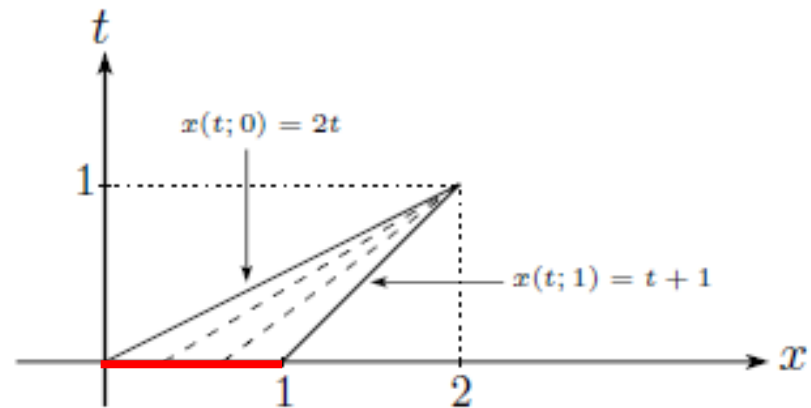
Consider the IVP: $u_t + uu_x = 0$, $u(x,0) = f(x)$, $x \in \mathbb{R}$

$$f(x) = \begin{cases} 2, & x < 0 \\ 2 - x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$



Recall that $t_b = 1$ (and $x_b = 2$) and the solution reads:

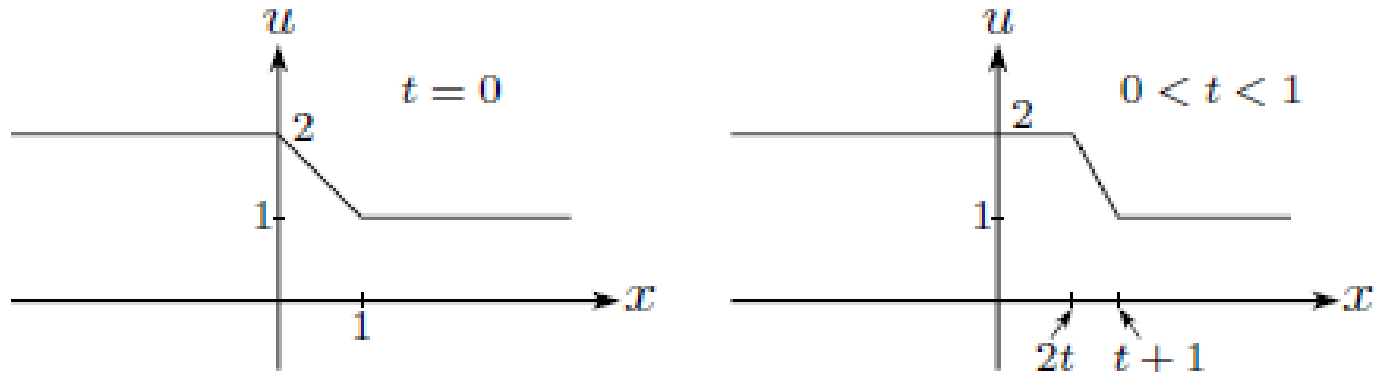
$$u(x,t) = \begin{cases} 2, & x < 2t \\ \frac{2-x}{1-t}, & 2t \leq x \leq t+1 \\ 1, & x > t+1 \end{cases}$$



Hence, at $t = t_b = 1$ the solution becomes **discontinuous**

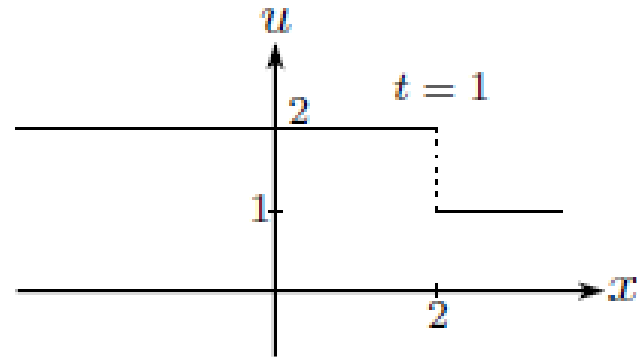
Revisiting a previous example – cont.

In particular, the solution has the form:



and for $t = t_b = 1$ the solution reads:

$$u(x, t = 1) = \begin{cases} 2, & x < 2 \\ 1, & x > 2 \end{cases}$$



Hence, for $t > t_b = 1$, **no classical solution exists**

Nevertheless, we can construct a **weak solution** as follows


Constructing the weak solution

Observe that:
$$u = \begin{cases} u^- = 2, & \text{for } (x, t) \text{ on the left of } s(t) \\ u^+ = 1 & \text{for } (x, t) \text{ on the right of } s(t) \end{cases}$$

with $s(t)$ satisfying the ODE:
$$\frac{ds}{dt} = \frac{F(u^+, t) - F(u^-, t)}{u^+ - u^-}, \quad s(t_b) = x_b$$

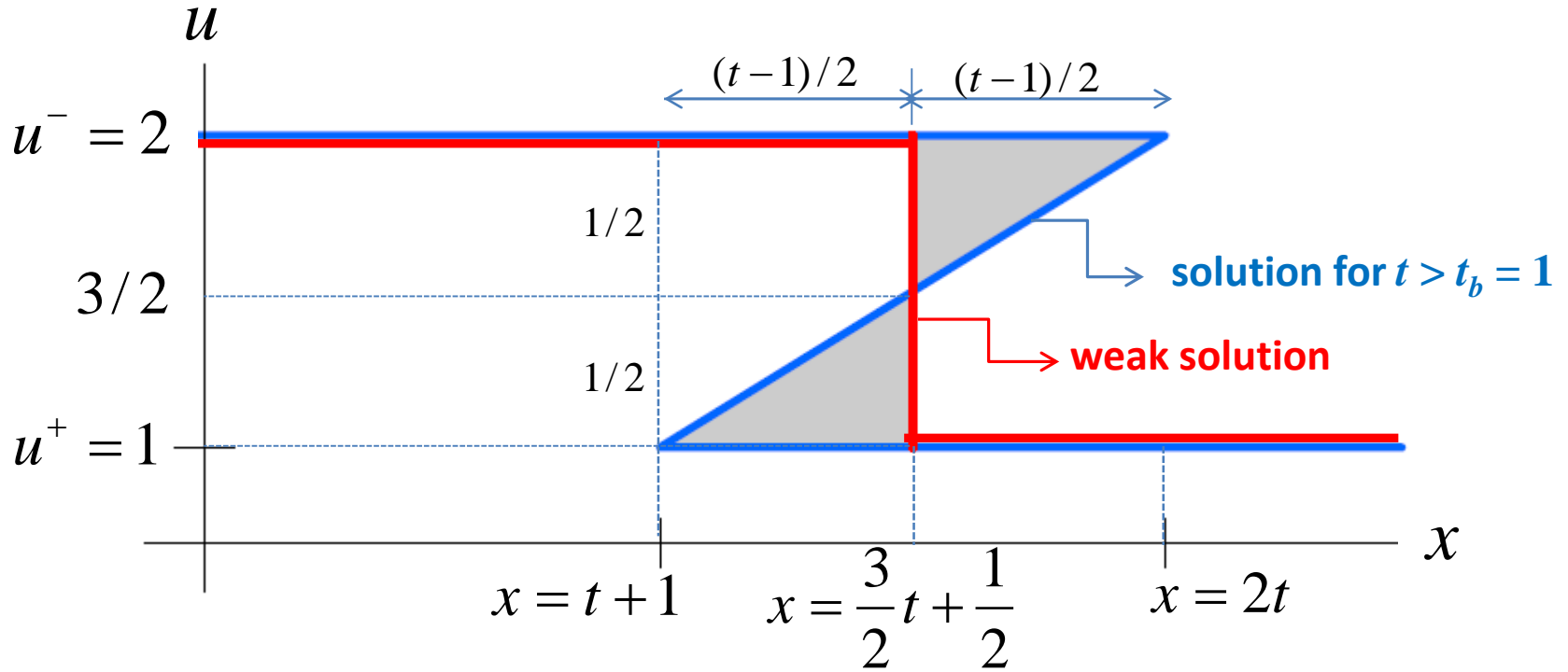
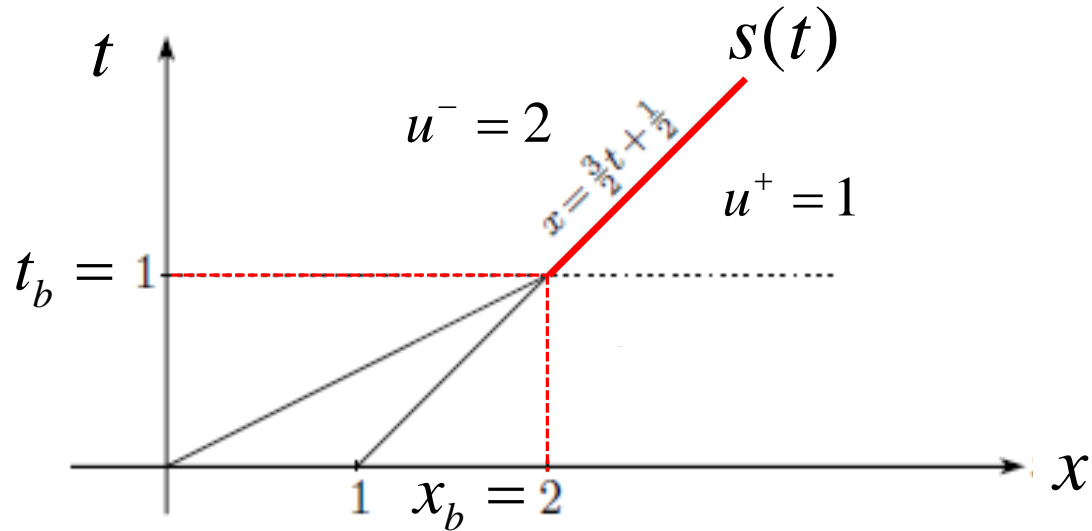
Here:
$$F(u) = \frac{1}{2}u^2$$

$$\left. \begin{aligned} \frac{ds}{dt} &= \frac{F(u^+, t) - F(u^-, t)}{u^+ - u^-} \\ \frac{ds}{dt} &= \frac{1}{2}(u^+ + u^-) \end{aligned} \right\} \Rightarrow \frac{ds}{dt} = \frac{1}{2}(u^+ + u^-) = \frac{1}{2}(1 + 2) = \frac{3}{2}$$


**RH condition
for the usual Hopf equation**

Hence:
$$\left. \begin{aligned} \frac{ds}{dt} &= \frac{3}{2} \Rightarrow s(t) = \frac{3}{2}t + x_0 \\ s(t_b) &= x_b \Rightarrow s(1) = 2 \end{aligned} \right\} \Rightarrow s(t) = \frac{3}{2}t + \frac{1}{2}$$

The xt -plane and the weak solution



The generalized solution

■ Concluding, a **generalized solution** of the considered IVP is:

$$u(x, t) = \begin{cases} u_{classical}(x, t), & \text{for } t < t_b = 1 \\ u_{weak}(x, t) & \text{for } t \geq t_b = 1 \end{cases}$$

■ The **classical solution** for $t < 1$ reads:

$$u_{classical}(x, t) = \begin{cases} 2, & x < 2t \\ \frac{2-x}{1-t}, & 2t \leq x \leq t+1 \\ 1, & x > t+1 \end{cases}$$

■ The **weak solution** for $t \geq 1$ reads:

$$u_{weak}(x, t) = \begin{cases} 2, & \text{for } x < \frac{3}{2}t + \frac{1}{2} \\ 1, & \text{for } x > \frac{3}{2}t + \frac{1}{2} \end{cases}$$

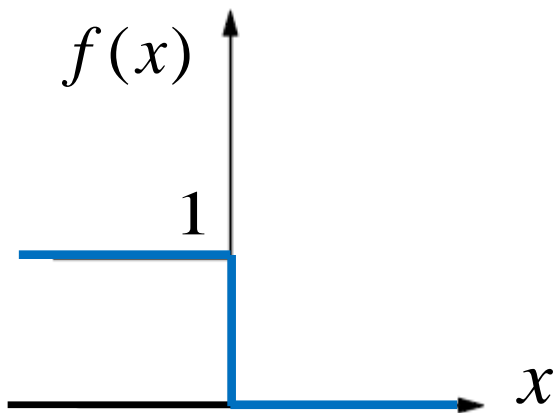
Riemann problems

- A **Riemann problem** is an initial value problem for a hyperbolic PDE (or a system thereof) in which the initial data is **piecewise constant with a discontinuity**

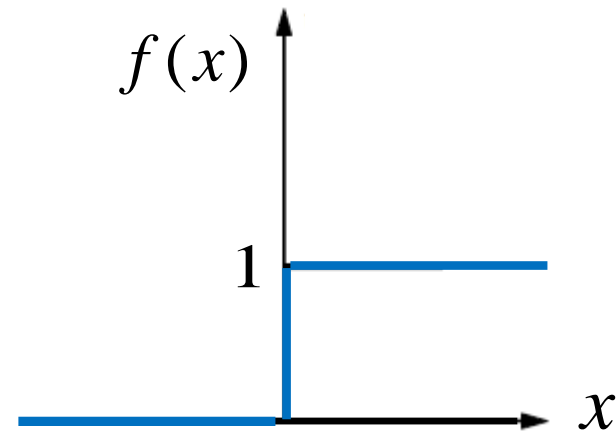
Examples

Consider the IVP: $u_t + u u_x = 0$, $u(x,0) = f(x)$, $x \in \mathbb{R}$, $t > 0$ with the following sets of discontinuous initial data:

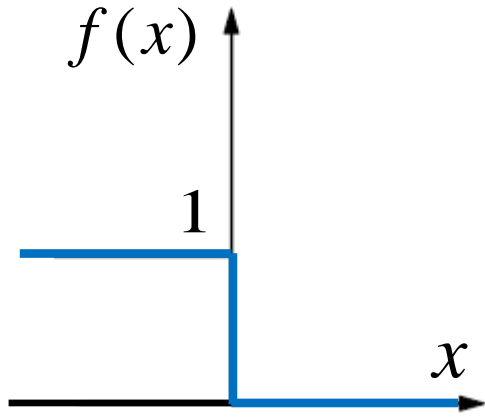
$$(I) \quad f(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$$



$$(II) \quad f(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$



Riemann problem (I)

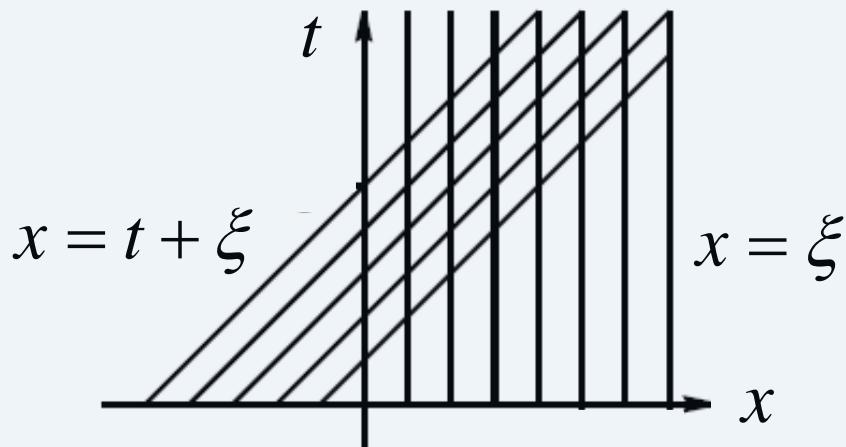


In this case, the initial data is already the form of a shock wave. Indeed:

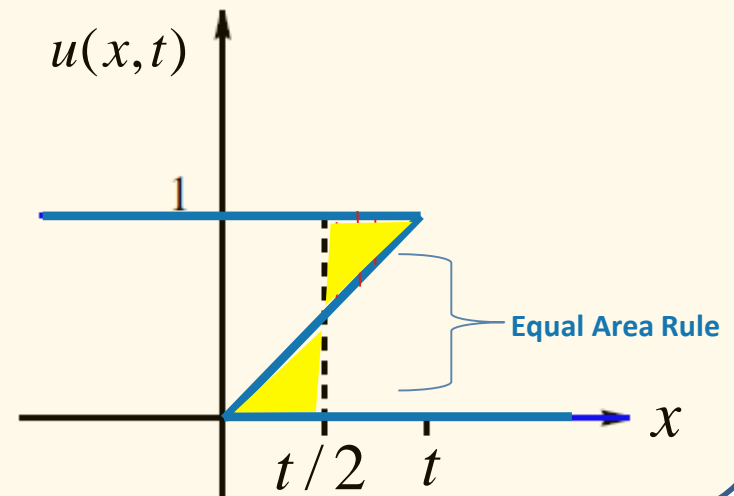
$$\frac{dx}{dt} = u, \quad x(0) = \xi; \quad \frac{du}{dt} = 0, \quad u(0) = f(\xi)$$

lead to: $x = f(\xi)t + \xi; \quad u(x,t) = f(\xi) = \begin{cases} 1, & \xi < 0 \text{ or } x < t \\ 0, & \xi > 0 \text{ or } x > 0 \end{cases}$

Characteristics intersect for $0 < x < t$



Form of $u(x,t)$



Riemann problem (I) – cont.

Weak solution: $u = \begin{cases} u^- = 1, & \text{for } (x,t) \text{ on the left of } s(t) \\ u^+ = 0 & \text{for } (x,t) \text{ on the right of } s(t) \end{cases}$

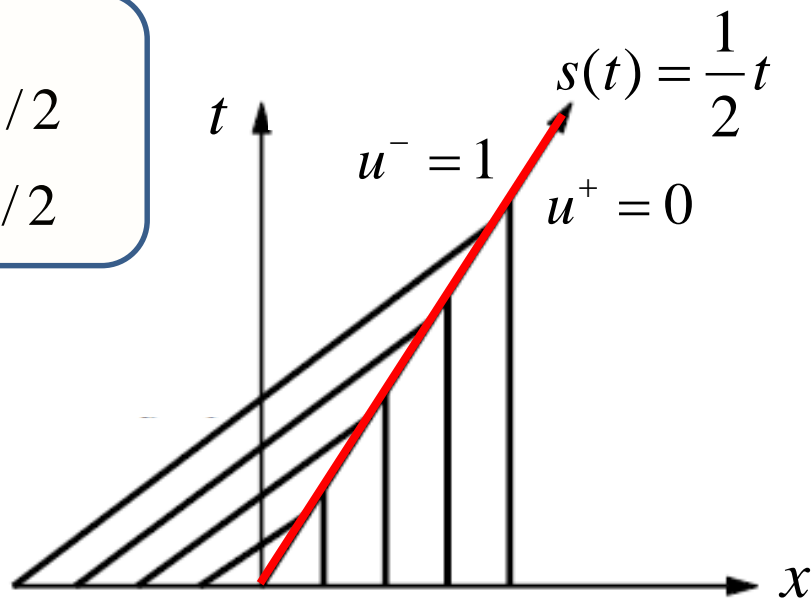
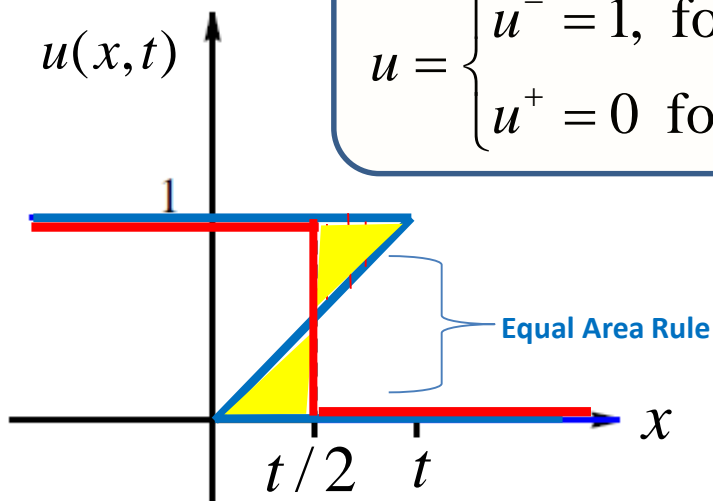
with $s(t)$ satisfying the ODE: $\frac{ds}{dt} = \frac{1}{2}(u^+ + u^-)$, $s(0) = 0$

RH condition

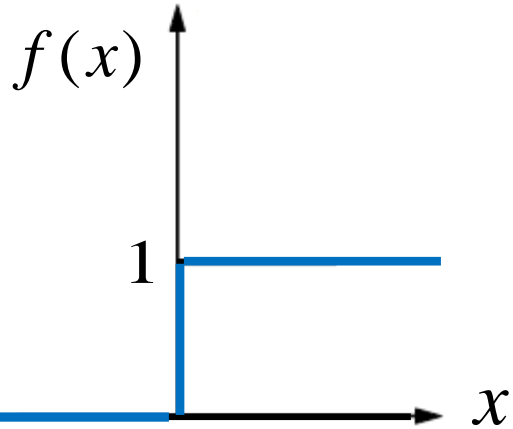
$$\frac{ds}{dt} = \frac{1}{2}(u^+ + u^-) = \frac{1}{2}(0 + 1) = \frac{1}{2} \Rightarrow s(t) = \frac{1}{2}t$$

weak solution

$$u = \begin{cases} u^- = 1, & \text{for } x < t/2 \\ u^+ = 0 & \text{for } x > t/2 \end{cases}$$



Riemann problem (II)



In this case, we again have:

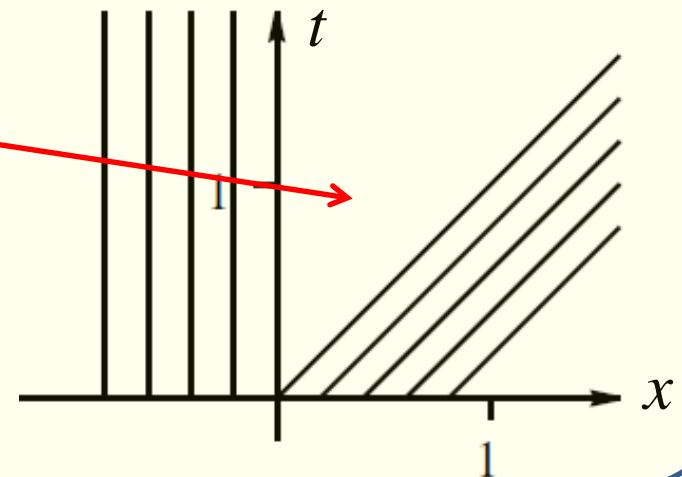
$$\frac{dx}{dt} = u, \quad x(0) = \xi; \quad \frac{du}{dt} = 0, \quad u(0) = f(\xi)$$

and the characteristics are of the form:

$$x = f(\xi)t + \xi; \quad u(x,t) = f(\xi) = \begin{cases} 0, & \xi < 0 \\ 1, & \xi > 0 \end{cases} \Rightarrow x = \begin{cases} \xi, & \xi < 0 \\ t + \xi, & \xi > 0 \end{cases}$$

However, we still have a problem: there exists **a region** on which there is not enough information!

No crossing of characteristics



Question:

How should we define the solution in this region?

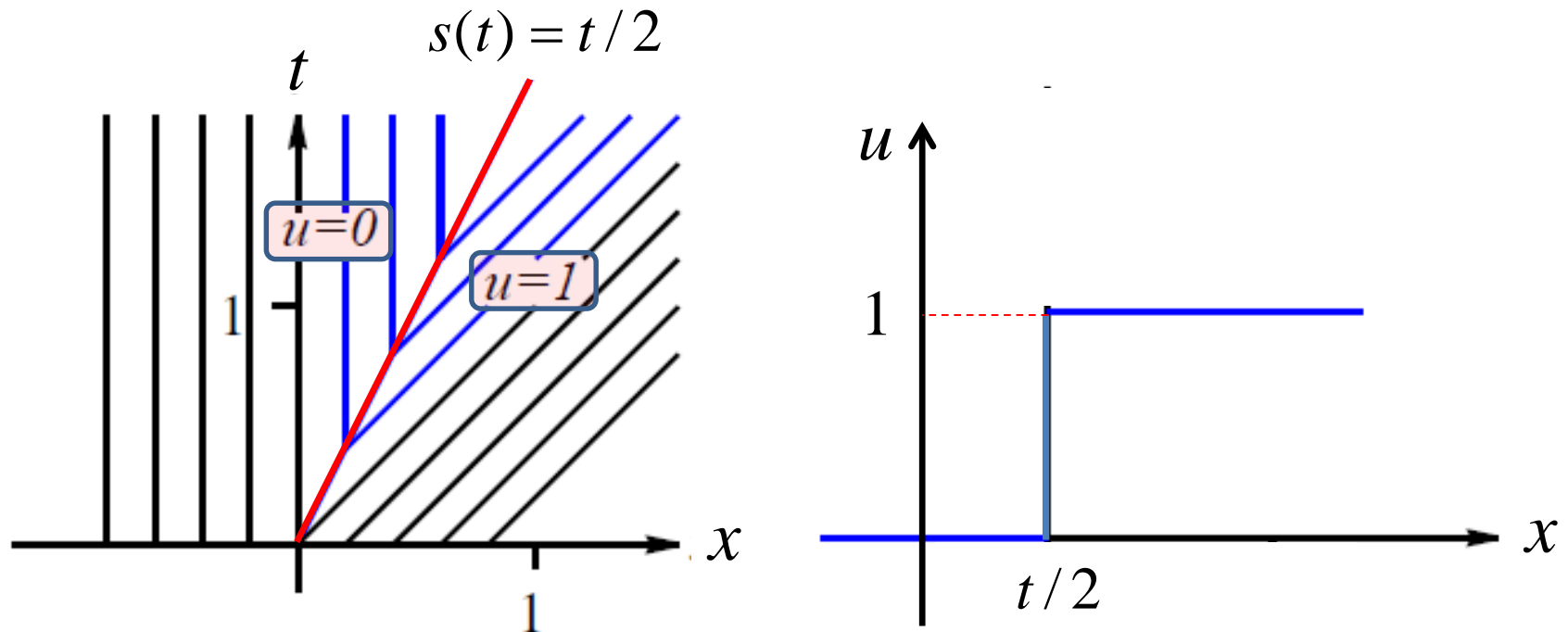
Riemann problem (II) – cont.

One possibility:

A possible **weak solution** of the problem is:

$$u = \begin{cases} 0, & \text{for } x < t/2 \\ 1, & \text{for } x > t/2 \end{cases}$$

This is composed by classical solutions on either side of the **curve of discontinuity** $s(t)=t/2$. Furthermore, this solution satisfies the **RH condition** along the curve of discontinuity.



Riemann problem (II) – cont.

Another possibility:

Another possible **weak solution** of the problem can be found upon **filling the wedge** $0 < x < t$ with another, **similarity solution**, of the Hopf equation. Such a **similarity solution** is of the form:

$$u(x, t) = t^n g(\eta), \quad \eta = xt^m$$

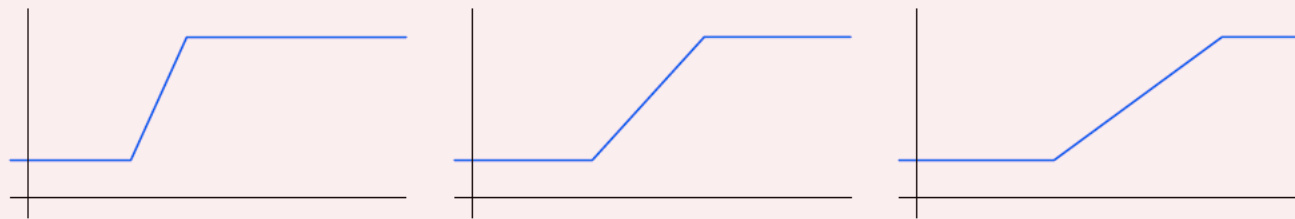
Observing that the Hopf equation $u_t + uu_x = 0$ is invariant under the transformations $x \mapsto \lambda x$, $t \mapsto \lambda t$, a simple calculation leads to:

$$n = 0, \quad m = -1, \quad g = \frac{x}{t}$$

Hence, another solution of the problem is the continuous function:

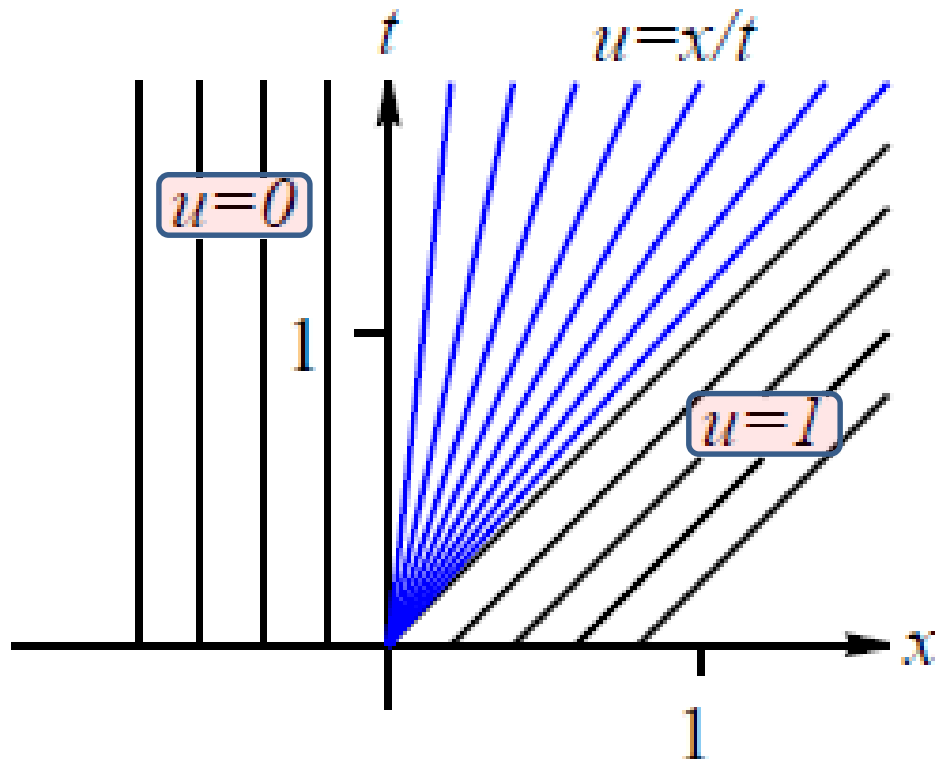
$$u = \begin{cases} 0, & \text{for } x < 0 \\ \frac{x}{t}, & \text{for } 0 < x < t \\ 1, & \text{for } x > t \end{cases}$$

This **weak solution** is called a **rarefaction wave**



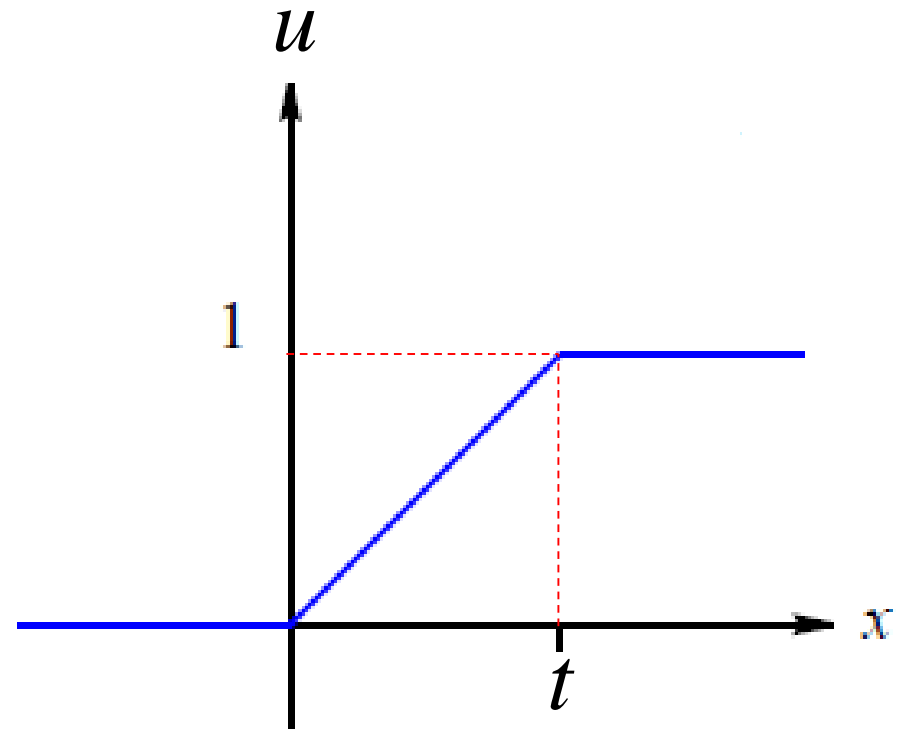
Riemann problem (II) – cont.

xt -plane and characteristics



Notice the **“fan”** at $x = 0$

The form of the solution



Recall:
$$u = \begin{cases} 0, & \text{for } x < 0 \\ \frac{x}{t}, & \text{for } 0 < x < t \\ 1, & \text{for } x > t \end{cases}$$

The Lax entropy condition

There exist at least **two solutions** of the Riemann problem (II), with the same initial data and hence **the solution is not unique**.

Question: **which one of these solutions is physically meaningful?**

For our initial data, **the wave is higher to the right**. Consequently, we expect **the part of the wave to the right to move faster**. Hence, physically, we do not want to allow for the 1st solution. Instead, **we accept the 2nd one as a physically more realistic solution**.

It can be proved that there exists a unique weak discontinuous solution of the Cauchy problem which satisfies the following inequality on the curve of discontinuity:

$$c(u^-) > s'(t) > c(u^+)$$

Lax entropy condition

Criterion for a unique weak discontinuous solution

The Lax entropy condition – cont.

$$c(u^-) > s'(t) > c(u^+)$$

- This condition states that **the wave speed just behind the shock is greater than the wave speed just ahead of it**. In other words, **the wave behind the shock catches up to the wave ahead of it**.
- This entropy criterion is a special case of the **second law of thermodynamics: entropy increases across a shock**.
- **Geometrically**, Lax entropy condition can be stated as follows: The characteristics originating on either side of the discontinuity curve, when continued in the direction of increasing t , intersect the curve of discontinuity.
- *For every nonlinear IVP there exists a unique weak solution defined $\forall t \geq 0$ with only shock as a discontinuity.* The **proof** is fairly difficult and was provided by **Lax (1973)**.