The Korteweg-de Vries – Burgers (KdV-B) equation

A model for dispersive shock waves

Introducing the Korteweg-de Vries – Burgers (KdVB) equation

$$\boxed{\underbrace{u_t + uu_x}_{\text{Hopf equation}} + \underbrace{\mu u_{xxx}}_{\text{dispersion}} - \underbrace{\nu u_{xx}}_{\text{diffusion}} = 0}$$

- For $\mu = \nu = 0$, i.e., in the absence of dispersion and diffusion, the KdVB becomes **the Hopf equation**, a prototypical quasi-linear PDE for the study of **shock waves**

- For $\mu = 0$ **and** $\nu \neq 0$, the KdVB becomes **the Burgers equation**; this model supports **monotone viscous shock waves**

- For $\nu = 0$ **and** $\mu \neq 0$, the KdVB becomes **the KdV equation**; this model supports **solitons**

- For $\mu = 0$ **and in the absence of nonlinearity**, the KdVB becomes **the linear diffusion equation**

Traveling wave solutions

We seek **traveling wave solutions** of the KdVB equation

$$\boxed{u_t + uu_x + \mu u_{xxx} - \nu u_{xx} = 0}$$

of the form: $\boxed{u(x,t) = u(\xi),}$ $\boxed{\xi = x - ct,}$

and derive the following 3d-order ODE:

$$\underline{-cu' + uu' + \mu u''' - \nu u'' = 0.}$$

Then, integrating with respect to ξ we obtain:

$$\boxed{\mu u'' = cu - \frac{1}{2}u^2 + \nu u' + K}$$

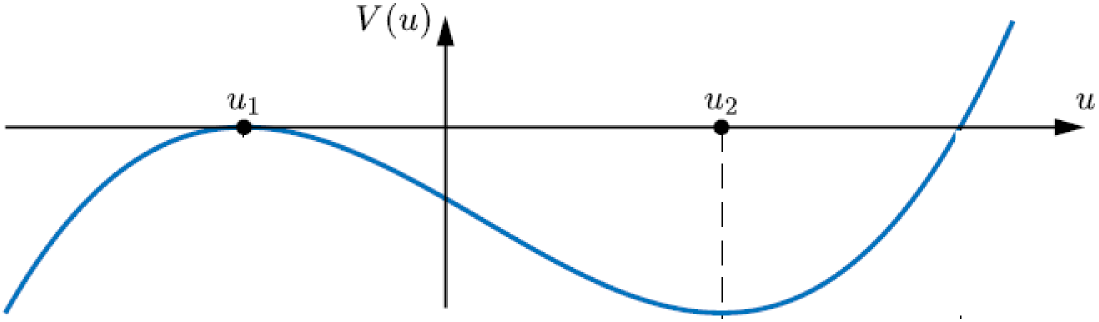
where K is a constant of integration

The associated dynamical system

$$\mu u'' = cu - \frac{1}{2}u^2 + \nu u' + K$$

In the absence of diffusion ($\nu = 0$), this ODE can be viewed as **the equation of motion of a particle in the presence of the potential:**

$$V(u) = \frac{1}{6}u^3 - \frac{1}{2}cu^2 - Ku + C$$



In the presence of diffusion ($\nu \neq 0$), this ODE can be written as**:**

$$\frac{d}{d\xi}\left[\frac{1}{2}\mu\left(\frac{du}{d\xi}\right)^2 + V(u)\right] = f_{\text{fr}}\frac{du}{d\xi},$$

$f_{\text{fr}} = \nu u'$ **negative friction**

The system of 1st-order ODEs

Introducing the "velocity" $\boxed{v = u'}$ we rewrite the 2nd-order ODE:

$$\mu u'' = cu - \frac{1}{2}u^2 + \nu u' + K$$

as system of 1st-order ODEs:

$$\boxed{\begin{aligned} u' &= v, \\ v' &= \frac{\nu}{\mu}v - \frac{1}{2\mu}(u - u_1)(u - u_2) \end{aligned}}$$

Here, $u_{1,2}$ are the roots **of the quadratic polynomial**

$\underline{-(1/2)u^2 + cu + K = 0}$, **assumed to be real**. The roots are:

$$\boxed{u_1 = c - \sqrt{c^2 + 2K}, \quad u_2 = c + \sqrt{c^2 + 2K},}$$

We thus require that: $\underline{c^2 + 2K > 0}$, and thus: $\underline{\underline{u_1 < u_2.}}$

Fixed points of the system

The system:

$$\begin{aligned} u' &= v, \\ v' &= \frac{\nu}{\mu} v - \frac{1}{2\mu}(u - u_1)(u - u_2) \end{aligned}$$

has the **fixed points:** $(u,\ v) = (u_1,\ 0),$ $(u,\ v) = (u_2,\ 0)$

We are interested in finding **solutions** of the system **with end states the above fixed points**, so that the solution curve:

$$\frac{dv}{du} = \frac{1}{2\mu v}\left[2\nu v - (u - u_1)(u - u_2)\right]$$

connects one fixed point to the other.

To find such solutions, we have to **investigate the stability of the fixed points**, following basic ODE theory.

Linearization around the fixed points

Consider the linearization ansatz: $\underline{u = u_* + \boxed{\tilde{u},}}$ $\underline{v = 0 + \boxed{\tilde{v}.}}$

small perturbations

where $\begin{cases} \underline{u_* = u_1} \text{ for the fixed point } (u_1, 0) \\ \underline{u_* = u_2} \text{ for the fixed point } (u_2, 0) \end{cases}$

Substituting, and **retaining only the linear terms** in the expansion of the right-hand sides, we derive the linearized system:

$$\boxed{\begin{aligned} \tilde{u}' &= \tilde{U}(\tilde{u}, \tilde{v}) \equiv \tilde{v}, \\ \tilde{v}' &= \tilde{V}(\tilde{u}, \tilde{v}) \equiv \frac{\nu}{\mu}\tilde{v} + s\frac{1}{\mu}\sqrt{c^2 + 2K}\ \tilde{u}. \end{aligned}}$$

where $s = +1$ for $(u_1, 0)$ or $s = -1$ for $(u_2, 0)$.

Then, we evaluate the relevant **Jacobian matrix *J* at the equilibrium points** $(u_*, 0)$

Classification of the fixed points (I)

Jacobian matrix:

$$J = \begin{pmatrix} \partial\tilde{U}/\partial\tilde{u} & \partial\tilde{U}/\partial\tilde{v} \\ \partial\tilde{V}/\partial\tilde{u} & \partial\tilde{V}/\partial\tilde{v} \end{pmatrix}\Bigg|_{(u_*,0)} = \begin{pmatrix} 0 & 1 \\ (s/\mu)\sqrt{c^2+2K} & \nu/\mu \end{pmatrix}$$

Eigenvalues λ from the equation: $\underline{\det(J-\lambda I)=0}$

$$\boxed{\lambda_\pm = \frac{1}{2\mu}\left(\nu \pm \sqrt{\nu^2 + 4s\mu\sqrt{c^2+2K}}\right)}$$

We can now find the following:

For $s=+1$ we always have: $\underline{\lambda_\pm \in \mathbb{R}}$ and $\underline{\lambda_- < 0 < \lambda_+}$

Real eigenvalues of *opposite signs*

Hence: $(u_1, 0)$ is always a *saddle* point

Classification of the fixed points (II)

Furthermore, for $s = -1$ and since:

$$\lambda_{\pm} = \frac{1}{2\mu}\left(\nu \pm \sqrt{\nu^2 + 4s\mu\sqrt{c^2 + 2K}}\right)$$

we have the following cases:

$$\begin{aligned}
&\text{(a) } \lambda_{\pm} \in \mathbb{R}_{-} \quad \text{if} \quad \nu^2 > 4\mu\sqrt{c^2 + 2K}, \\
&\text{(b) } \lambda_{\pm} \in \mathbb{C} \text{ (complex conjugates)} \quad \text{if} \quad 0 < \nu^2 < 4\mu\sqrt{c^2 + 2K}, \\
&\text{(c) } \lambda_{\pm} \in \mathbb{I} \text{ (complex conjugates)} \quad \text{if} \quad \nu = 0,
\end{aligned}$$

Hence:

$$(u_2, 0) \quad \text{is} \quad \begin{cases} \text{a } \textit{stable node} & \text{if} \quad \nu^2 \geq 4\mu\sqrt{c^2 + 2K}, \\ \text{a } \textit{unstable spiral} & \text{if} \quad 0 < \nu^2 < 4\mu\sqrt{c^2 + 2K}, \\ \text{a } \textit{center} & \text{if} \quad \nu = 0. \end{cases}$$

Dissipation vs. dispersion

According to the above analysis, the **connection between the fixed points** depends on the **competition** (i.e., the relative strength) between **dissipation** and **dispersion**:

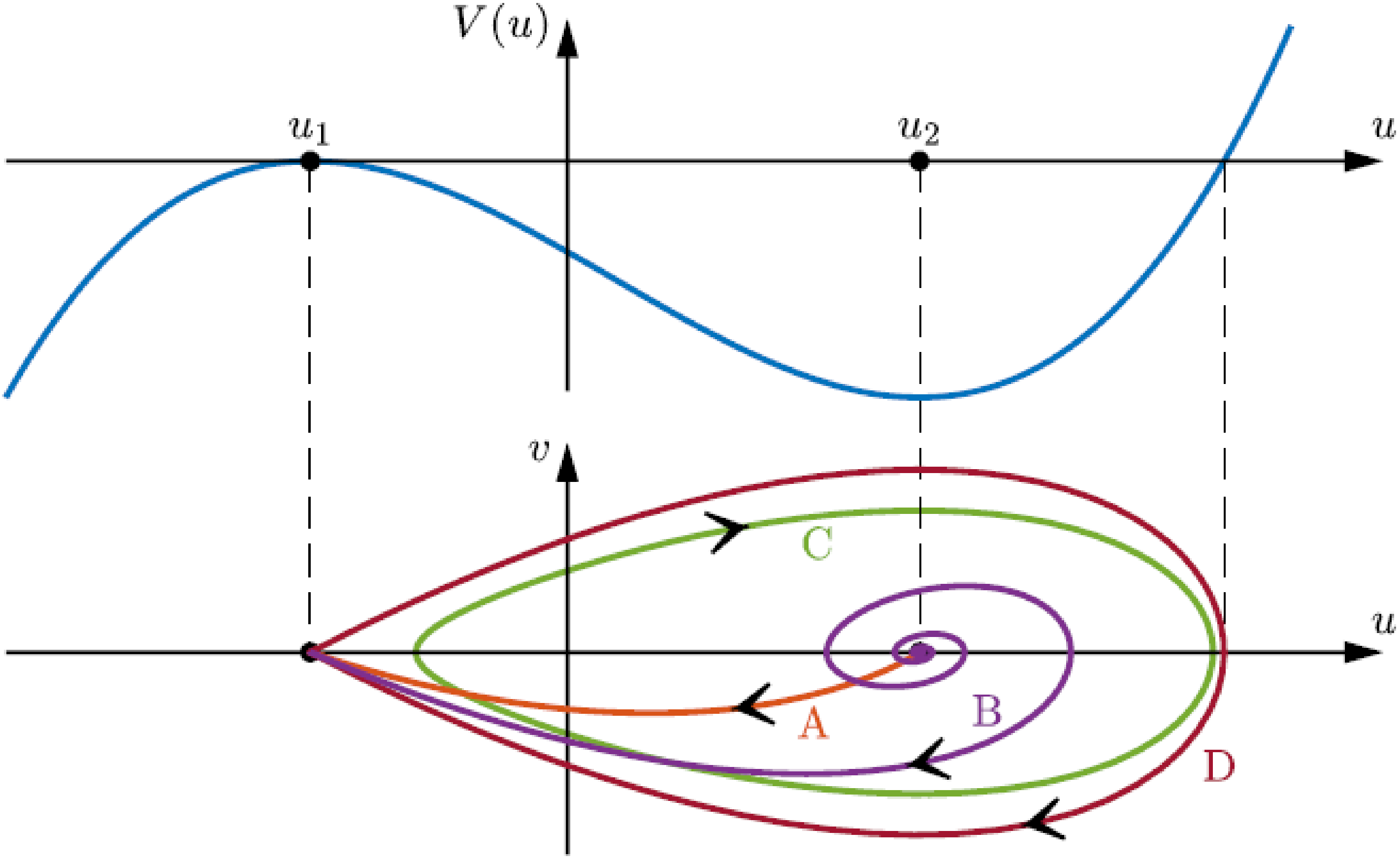
- If dissipation dominates, i.e., for $\nu^2 \geq 4\mu\sqrt{c^2+2K}$, under the action of the **negative friction**, the trajectory of the effective particle ascends **from the bottom**, at the fixed point $(u_2, 0)$, **to the top of the potential hill**, at $(u_1, 0)$.

In this case, the solution has the form of a **monotone shock wave**

- If dispersion dominates, i.e., for $0 < \nu^2 < 4\mu\sqrt{c^2+2K}$, the trajectory of the effective particle ascends **from the bottom**, at the fixed point $(u_2, 0)$, **to the top of the potential hill**, at $(u_1, 0)$, in an **oscillatory manner.**

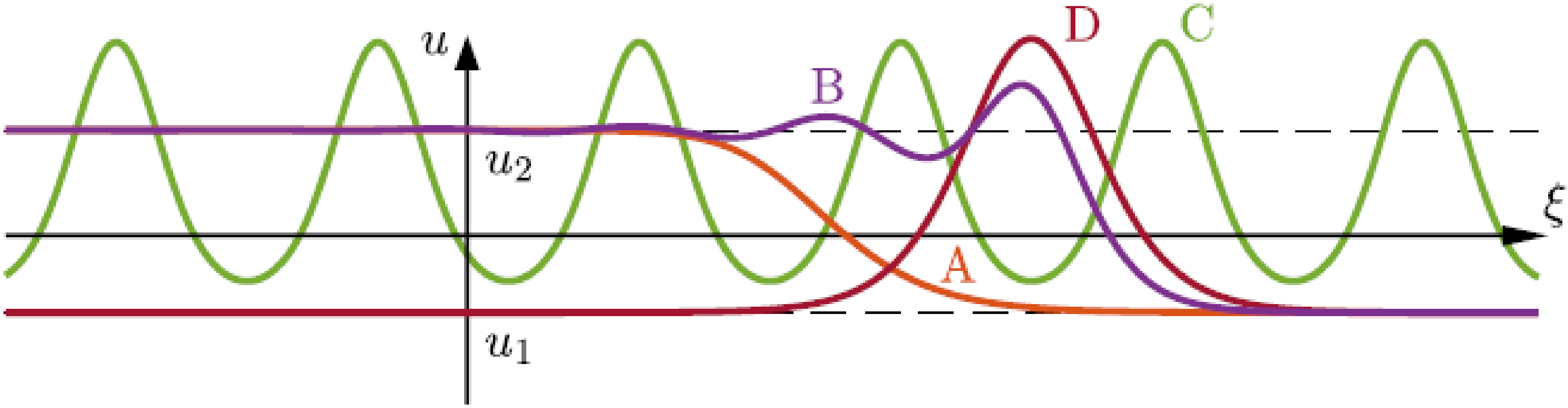
In this case, the solution has the form of a **oscillatory shock wave**

Monotone vs. oscillating shock waves



- **Orange curve (A)**: **monotone shock wave**, for $\nu^2 \geq 4\mu\sqrt{c^2+2K}$
- **Purple curve (B)**: **oscillating shock wave**, for $0 < \nu^2 < 4\mu\sqrt{c^2+2K}$
- **Green (C)** and **dark red (D)** curves: **cnoidal wave** and **soliton** of the **KdV equation** for $\nu = 0$

The form of the possible solutions



- **Curve (A):** monotone shock wave, $\nu^2 \geq 4\mu\sqrt{c^2 + 2K}$
- **Curve (B):** oscillatory shock wave, $0 < \nu^2 < 4\mu\sqrt{c^2 + 2K}$
- **Curve (C):** cnoidal wave of the KdV equation, $\nu = 0$
- **Curve (D):** KdV soliton, $\nu = 0$

- The oscillatory shock wave may, in fact, be regarded as a **combination** of a **KdV soliton** and a **damped cnoidal wave**.

Oscillatory shock waves of the KdV-Burgers equation may be used as a prototype for the description of **undular bores**

An example of an undular bore

Three surfers riding a tidal bore at Turnagain Arm, Alaska