KdV: $u_t + 6uu_x + u_{xxx} = 0$ (1)

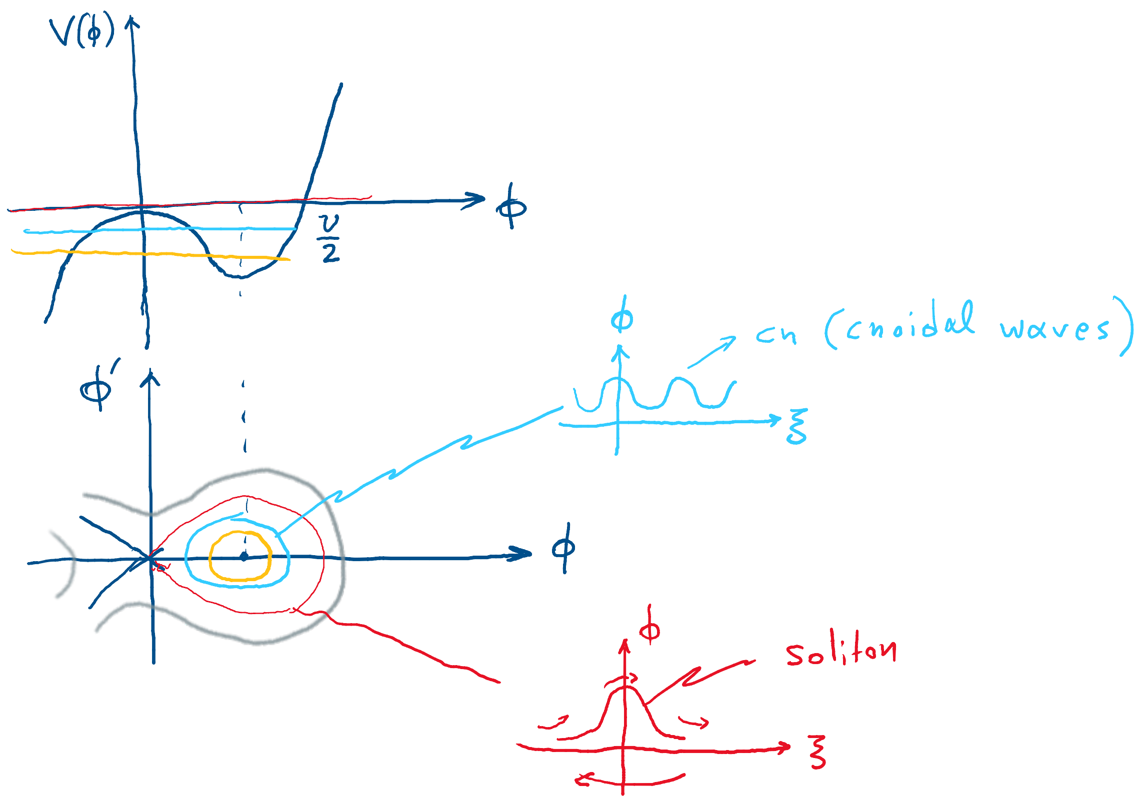
$u, u_x, \ldots \longrightarrow 0$ για $x \to \pm\infty$

Λύσεις της μορφής: $u = \phi(\xi)$; $\xi = x - vt$ (2)

(1): $-v\phi' + 6\phi\phi' + \phi''' = 0 \Rightarrow$

$$\phi'' - v\phi + 3\phi^2 = 0 \Rightarrow \phi'' = v\phi - 3\phi^2 = -\frac{\partial V}{\partial \phi} \quad (3)$$

(3): $V(\phi) = -\frac{v}{2}\phi^2 + \phi^3$



(3): $\phi'' - v\phi + 3\phi^2 = 0 \Rightarrow$

$$\rightarrow \frac{1}{2}\phi'^2 \underbrace{- \frac{v}{2}\phi^2 + \phi^3}_{V(\phi)} = h \quad (4)$$

Για $\underline{h = 0}$ έχουμε τη λύση του σολιτονίου

$$(4): \quad \int \frac{d\phi}{\sqrt{v\phi^2-2\phi^3}} = \pm\int d\xi \qquad (5)$$

Γνωρίζουμε: $\displaystyle\int \frac{dw}{w\sqrt{1-w^2}} = \operatorname{sech}^{-1} w$

Θέτουμε $\phi = Aw^2 \Rightarrow d\phi = 2Aw\,dw$

$$\frac{d\phi}{\sqrt{v\phi^2-2\phi^3}} = \frac{d\phi}{\phi\sqrt{v-2\phi}} = \frac{2\cancel{A}\cancel{w}\,dw}{\cancel{A}w^{\cancel{2}}\sqrt{v-2Aw^2}} =$$

$$= \frac{2\,dw}{\sqrt{v}\,w\sqrt{1-\frac{2A}{v}w^2}}$$

και για $\frac{2A}{v} = 1 \Rightarrow A = \frac{v}{2}$ είναι τελικά:

$$\left.\int \frac{d\phi}{\sqrt{v\phi^2-2\phi^3}} = \underbrace{\frac{2}{\sqrt{v}}\int \frac{dw}{w\sqrt{1-w^2}}}_{\frac{2}{\sqrt{v}}\operatorname{sech}^{-1} w} = \pm(\xi - x_0)\right\} \Rightarrow$$

$$\left.\begin{aligned} \Rightarrow\; & w = \operatorname{sech}\left[\frac{\sqrt{v}}{2}(\xi - x_0)\right] \\ & \phi = Aw^2 = \frac{v}{2}w^2 \end{aligned}\right\} \Rightarrow \phi = \frac{v}{2}\operatorname{sech}^2\left[\frac{\sqrt{v}}{2}(\xi - x_0)\right]$$

Θέτουμε $2\kappa^2 = \frac{v}{2} \Rightarrow v = 4\kappa^2$ οπότε το σολιτόνιο της KdV παίρνει τη μορφή:

$$\boxed{u(x,t) = 2\kappa^2\operatorname{sech}^2\left[\kappa(x - 4\kappa t - x_0)\right]}$$